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GEOMETRIC ASPECTS OF THE LINEAR COMPLEMENTARITY PROBLEM.(U)

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by

Richard E. Stone

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Geometric Aspects of the Linear Complementarity Problem

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ABSTRACT

A large part of the study of the Linear Complementarity Problem (**LCP**) has been concerned with matrix classes. A classic result of Samelson, Thrall, and Wesler is that the real square matrices with positive principal minors (**P**-matrices) are exactly those matrices M for which the **LCP** (q, M) has a unique solution for all real vectors q. Taking this geometrical characterization of the **P**-matrices and weakening, in an appropriate manner, some of the conditions, we obtain and study other useful and broad matrix classes thus enhancing our understanding of the **LCP**.

In Chapter 2, we consider a generalization of the **P**-matrices by defining the class **U** as all real square matrices M where, if for all vectors x within some open ball around the vector q the **LCP** (x, M) has a solution, then (q, M) has a unique solution. We develop a characterization of **U** along with more specialized conditions on a matrix for sufficiency or necessity of being in **U**.

Chapter 3 is concerned with the introduction and characterization of the class **INS**. The class **INS** is a generalization of **U** gotten by requiring that the appropriate **LCP**'s (q, M) have exactly k solutions, for some positive integer k depending only on M. Hence, **U** is exactly those matrices belonging to **INS** with k equal to one.

Chapter 4 continues the study of the matrices in **INS**. The range of values for k, the set of q where (q, M) does not have k solutions, and the multiple partitioning structure of the complementary cones associated with the problem are central topics discussed.

Chapter 5 discusses these new classes in light of known **LCP** theory, and reviews its better known matrix classes.

Chapter 6 considers some problems which remain open.

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CHAPTER 1.

BACKGROUND TO THE LINEAR COMPLEMENTARITY PROBLEM

1.1 Introduction

The central topic with which this work is concerned is the linear complementarity problem (LCP). The LCP is a nonlinear system of inequalities where we are given as data an $n \times n$ real matrix M, a real n-vector q, and are asked to find a real n-vector z such that

$$z \geq 0, \tag{1.1}$$

$$Mz + q \geq 0, \tag{1.2}$$

$$z^T(Mz + q) = 0. \tag{1.3}$$

Although we shall not do so here, one can consider the more general complementarity problem: given a closed convex cone $K \subseteq \Re^n$, with positive polar cone $K^* = \{y \in \Re^n : y^T x \geq 0, \text{ for all } x \in K\}$, and a function $F: K \to \Re^n$, find $z \in \Re^n$ such that

$$z \in K, \tag{1.4}$$

$$F(z) \in K^*, \tag{1.5}$$

$$z^T F(z) = 0. \tag{1.6}$$

These problems may be thought of as the natural formulations to use in situations where an equilibrium point is being sought. They arise in quite a number of fields including engineering, economics, optimization, game theory and control theory. For more on these applications see, for example, Lemke and Howson (1964), Cottle and Dantzig (1968), Cohen (1975), Koehler (1979), and Cottle, Giannessi and Lions (1980).

As the previous references would suggest, the LCP has been extensively studied. Most of this research has emphasized the *algebraic* nature of the problem. In the present work we study the LCP from a *geometric* viewpoint. Other authors have also taken this direction, see Saigal (1970b, 1972a, 1972b), Murty (1972), Eaves (1979), Kelly and Watson (1979), Garcia and Gould (1980), Howe (1980), Cottle, von Randow, and Stone (1981), and Doverspike and Lemke (1981). This work studies the characterization and general properties of matrices M for which (q, M) has the same number of solutions "globally," and, as a special case, has a unique solution "globally." (Here "globally" is used to mean "for all $q \in \Re^n$ for which (q, M) has a solution, except possibly for a set of measure zero." This will be explained in more detail later.) Other works often are concerned with exhibiting algorithms that "process" the LCP for a specified matrix class, and then, possibly, using the algorithm to show various properties of that class. In this work we are concerned with existence proofs and properties of matrix classes rather than with algorithms. The questions studied do not seem to lend themselves to algorithmic techniques.

In Chapter 2 we will study LCP's which have either no solutions or unique solutions at almost every point. We will derive necessary and sufficient conditions for a matrix M to have the property that if for some $q_0 \in \Re^n$ and some $\epsilon > 0$ the LCP (1.1)–(1.3) has a solution for all $q \in \Re^n$ within

a distance of ϵ from q_0, then the LCP has precisely *one* solution for q_0. Further results on matrices where the related LCP has this "global" uniqueness property will be derived. A few papers show that some known matrix classes are of this type. We will examine these papers more closely in Chapter 5. In another direction, the question of local uniqueness in an LCP was studied by Mangasarian (1980). That paper exhibits necessary and sufficient conditions for a solution, z, to a given LCP to be within an open ball in $\Re^n$ that contains no other solutions to the LCP. Aside from keeping an algebraic outlook, these results are in a different vein from the questions we are presently considering and do not appear to be helpful to the current study.

In Chapters 3 and 4 we relax the condition of global uniqueness. In essence we replace the italicized word *"one"* in the previous paragraph with k, where k is some fixed positive integer. We will derive characterizations for these matrices and related results concerning the geometric structure of LCP's with this property. There are a few papers that deal with the property of an almost globally invariant number of solutions, see Murty (1972), Saigal (1972b), Kojima and Saigal (1979), and Mohan (1978, 1980). These papers deal with special matrix classes for which a specialized result is sought. They do not attack the problem in full generality, and some do not look for underlying geometric structure. Saigal (1972b) contains some errors – inherited by Mohan (1978) – which will be discussed in Chapter 5. These papers, along with some others, e.g., Saigal (1972a), discuss the property of an almost globally invariant *parity* in the number of solutions. That is, the number of solutions to (1.1)–(1.3) for a particular matrix, M, will be either odd or even, not both, for almost all q. This is a much weaker property than that of an invariant number of solutions, and will not be given much consideration here. The interested reader should see Saigal (1972a)

for a complete geometric characterization of LCP's with this constant parity property.

Chapters 5 and 6 discuss other matrix classes, related LCP theory and some open questions. It is typical for dissertations in this field to begin with one or two chapters reviewing the known classes of matrices and the history of the area. In this work it seemed better to leave this to a later chapter. It is Chapter 5 that contains such a summary.

The next section of this chapter will go over preliminary results that are needed throughout this work. The last section of this chapter is a glossary of the notation that is used. It is suggested that the reader first look over this last section to see the basic style of notation used. It should be pointed out that throughout this work the word *interior* is used to mean *relative interior.*

1.2 Background Material

As was stated before, the Linear Complementarity Problem is: Given $M \in \Re^{n \times n}$ and $q \in \Re^n$, find $z \in \Re^n$ such that

$$z \geq 0, \tag{1.1}$$

$$Mz + q \geq 0, \tag{1.2}$$

$$z^T(Mz + q) = 0. \tag{1.3}$$

The LCP with M and q as the data will be denoted as: (q, M). For our purposes it will be useful to define $w = Mz + q$. Thus we can express (q, M) as the problem, given $M \in \Re^{n \times n}$ and $q \in \Re^n$, of finding $z, w \in \Re^n$ such that

$$Iw - Mz = q\,, \tag{1.7}$$

$$z, w \geq 0\,, \tag{1.8}$$

$$z^T w = 0\,, \tag{1.9}$$

where I is the $n \times n$ identity matrix. This formulation of the problem makes it clear that we are just trying to find a nonnegative linear *combination of* the column vectors of I and $-M$ that equals q, where we may not "use" both $I_{\cdot i}$ and $-M_{\cdot i}$ for any $i \in \overline{n}$. This idea suggests making

DEFINITION 1.1 For $M \in \Re^{n \times n}$ and $\alpha \in (\overline{n})$ define $C_M(\alpha) \in \Re^{n \times n}$ as

$$C_M(\alpha)_{\cdot i} \quad = \quad \begin{cases} I_{\cdot i} & , \text{if } i \notin \alpha \\ -M_{\cdot i} & , \text{if } i \in \alpha \end{cases} \tag{1.10}$$

where the subscript M will be dropped when it is clear to which M we are referring. The $C_M(\alpha)$ are called the *complementary matrices* associated with M. There are 2^n such matrices, not necessary distinct.

Associated with each complementary matrix is the finite convex cone

$$\text{pos}\, C_M(\alpha) = \{\, y \in \Re^n : y = C_M(\alpha)\, x, \quad x \geq 0 \,\}\,.$$

The cone $\text{pos}\, C_M(\alpha)$ is called a *complementary cone* of the matrix M, and the subscript M is dropped when it is clear which M is meant. There are 2^n such cones, not necessarily geometrically distinct. Notice that two distinct complementary matrices may be associated with complementary cones that are geometrically identical. For example, the matrix

$$M = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \tag{1.11}$$

will have $\text{pos}\, C(\{\,1\,\})$ geometrically equal to $\text{pos}\, C(\overline{2})$, even though $C(\{\,1\,\})$ and $C(\overline{2})$ are distinct matrices.

If we have $\beta \in (\overline{n})$ such that

$$\dim[\operatorname{pos} C_M(\alpha)_{\cdot\beta}] = r$$

then $\operatorname{pos} C_M(\alpha)_{\cdot\beta}$ is referred to as a *r-dimensional facet* of the complementary cone $\operatorname{pos} C_M(\alpha)$. Furthermore, if $|\beta| = n - 1$ then $\operatorname{pos} C_M(\alpha)_{\cdot\beta}$ is referred to as a *face* of the complementary cone $\operatorname{pos} C_M(\alpha)$.

Let $\operatorname{sol}(q, M)$ be the set of ordered pairs, (w, z), of solutions to the LCP (q, M). If $(w, z) \in \operatorname{sol}(q, M)$ then, letting $x = w + z \geq 0$ and $\alpha = \operatorname{supp} z$, we have $C(\alpha)x = q$. Conversely, if we find for some $\alpha \in (\overline{n})$ that there is an $x \geq 0$ with $C(\alpha)x = q$ then with $z_\alpha = x_\alpha$, $z_{\hat{\alpha}} = 0$, $w_\alpha = 0$ and $w_{\hat{\alpha}} = x_{\hat{\alpha}}$, we have $(w, z) \in \operatorname{sol}(q, M)$. In this way, each solution, $(w, z) \in \operatorname{sol}(q, M)$, will be associated with at least one complementary cone of M. Also, in this way, each point in a complementary cone of M will be associated with at least one solution. We can now state

DEFINITION 1.2 For $M \in \Re^{n \times n}$ let

$$\mathrm{K}(M) = \bigcup_{\alpha \in (\overline{n})} \operatorname{pos} C_M(\alpha).$$

We then see from the previous discussion that

$$\mathrm{K}(M) = \{ q \in \Re^n : \operatorname{sol}(q, M) \neq \emptyset \}.$$

In Figure 1.1 we show the complementary cones for the matrix in (1.11). In Figures 1.2 and 1.3 we show the complementary cones, respectively, for the matrices (1.12) and (1.13), where

$$\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \qquad\qquad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

(1.12) (1.13)

In these diagrams the column vector $I_{\cdot i}$ is indicated by an i and the column vector $-M_{\cdot i}$ is indicated by an i'.

Each solution of (q, M) must be associated with at least one complementary cone containing q, and each complementary cone containing q must be associated with at least one solution of (q, M). However, the exact relationship between complementary cones and solutions is often not simple. For example, consider the problem with M given by (1.12) and $q = (1,1)^T$. Then q is contained in three complementary cones $\operatorname{pos} C(\emptyset)$, $\operatorname{pos} C(\{1\})$ and $\operatorname{pos} C(\overline{2})$. However, $|\mathrm{sol}(q, M)| = 2$; where the solution $(w, z) = (1,1,0,0)$ is associated with $\operatorname{pos} C(\emptyset)$, and the solution $(w, z) = (0,0,1,0)$ is associated with the other two cones. With M given by (1.13) and $q = (0,-1)^T$, we find q is contained in the complementary cones: $\operatorname{pos} C(\overline{2})$ associated with the solution $(w, z) = (0,0,1,0)$; $\operatorname{pos} C(\{2\})$ associated with the solution $(w, z) = (1,0,0,1)$; and $\operatorname{pos} C(\{1\})$ associated with the infinitely many solutions $(w, z) = (0, \theta, 1+\theta, 0)$, where θ ranges over all nonnegative reals. In the first case we have have more complementary cones containing q than solutions to (q, M); in the second case there are more solutions to (q, M) than there are complementary cones containing q.

To help in our discussion, we make the

DEFINITION 1.3 For $M \in \Re^{n \times n}$, we say the complementary cone $\operatorname{pos} C_M(\alpha)$ is *full* or *nondegenerate* if and only if $\det C_M(\alpha) \neq 0$; otherwise we say the cone is *degenerate*. Notice $\det C_M(\alpha) = (-1)^{|\alpha|} \det M_{\alpha\alpha}$. More over a complementary cone is full if and only if it has positive n-dimensional volume, and a complementary cone is full if and only if it is not contained in an $(n-1)$-dimensional hyperplane. In addition to the above, we say M itself is *nondegenerate* if for all $\alpha \in (\overline{n})$ the cone $\operatorname{pos} C_M(\alpha)$ is nondegenerate, i.e.,

all the principal minors of M are nonzero.

DEFINITION 1.4 For $M \in \Re^{n \times n}$, we say the degenerate complementary cone $\operatorname{pos} C_M(\alpha)$ is *strongly degenerate* if and only if there exists a $z \in \Re^n$ such that $0 \neq z \geq 0$ and $C_M(\alpha) z = 0$, i.e., if and only if for $q = 0$, which is in *every* complementary cone, we find (q, M) has a non-trivial solution $(w, z) \neq 0$ associated with $\operatorname{pos} C_M(\alpha)$. Otherwise we say the cone is *weakly degenerate*. We say M is weakly degenerate if *not* all of its complementary cones are nondegenerate, but none of the complementary cones are strongly degenerate. That is, M is weakly degenerate if it has a zero principal minor and $\operatorname{sol}(0, M) = \{(0,0)\}$.

DEFINITION 1.5 For $M \in \Re^{n \times n}$, we say $(w, z) \in \operatorname{sol}(q, M)$ is a *nondegenerate* solution if and only if $w + z > 0$. Otherwise the solution is said to be *degenerate*. We say a point $q \in \Re^n$ is *nondegenerate* with respect to M if all solutions to (q, M) are nondegenerate. Otherwise q is said to be *degenerate*.

Consider the matrix M as given by (1.11). M is a degenerate matrix as all of its complementary cones are strongly degenerate, except for the nondegenerate complementary cone $\operatorname{pos} C(\emptyset)$. The matrix M as given by (1.13) is degenerate as it contains the weakly degenerate complementary cone $\operatorname{pos} C(\{1\})$. Finally, let M be the nondegenerate matrix given by (1.12). If $q = (1,1)^T$, then q is degenerate as $(w, z) = (0,0,1,0)$ is a degenerate solution to (q, M). If $q = (3,1)^T$, then q is nondegenerate as $(w^1, z^1) = (3,1,0,0)$ and $(w^2, z^2) = (0,0,2,1)$ are the only solutions to (q, M) and both are nondegenerate. The reader should refer to Figures 1.1, 1.2, and 1.3, respectively, when considering the matrices given by (1.11), (1.12), and (1.13).

Now, suppose q is contained in the interior of the degenerate complementary cone $\text{pos}\, C_M(\alpha)$. Thus there is some $0 < x \in \Re^n$ such that $C(\alpha)x = q$. As this is a degenerate cone, there exists $0 \neq y \in \Re^n$ such that $C(\alpha)y = 0$. Thus we may select some real number $\lambda \neq 0$ such that $0 \not< x + \lambda y \geq 0$. Hence, if we let $z_\alpha = (x + \theta y)_\alpha$, $z_{\hat{\alpha}} = 0$, $w_\alpha = 0$, and $w_{\hat{\alpha}} = (x + \theta y)_{\hat{\alpha}}$, then (w, z) is a solution to (q, M) for all θ such that $|\theta| \leq |\lambda|$. Hence, if M is degenerate then we have some $q \in \Re$ with $|\text{sol}(q, M)| = \infty$. Notice also that (q, M) has a degenerate solution when we let $\theta = \lambda$. In fact, this holds even if we have q on the boundary of the degenerate complementary cone. However, now we might have $\lambda = 0$, and so we might not have infinitely many solutions, but we will still have a degenerate solution.

Suppose q is contained in the nondegenerate complementary cone $\text{pos}\, C_M(\alpha)$. Thus there is some $0 \leq x \in \Re^n$ with $C(\alpha)x = q$. But now $C(\alpha)^{-1}$ exists and $x = C(\alpha)^{-1}q$. So with $z_\alpha = x_\alpha$, $z_{\hat{\alpha}} = 0$, $w_\alpha = 0$, and $w_{\hat{\alpha}} = x_{\hat{\alpha}}$, we have $([w_\alpha, w_{\hat{\alpha}}], [z_\alpha, z_{\hat{\alpha}}])$ is the only solution to (q, M) associated with the complementary cone $\text{pos}\, C_M(\alpha)$. In fact, if the solution (w, z) is nondegenerate, i.e., if $z_\alpha > 0$ and $w_{\hat{\alpha}} > 0$, then this solution is associated with no other complementary cone. For if it were associated with $\text{pos}\, C_M(\beta)$, $\beta \in (\overline{n})$, then we would need $z_{\hat{\beta}} = 0$ and $w_\beta = 0$ which, with the previous, would imply $\alpha = \beta$. We now have

PROPOSITION 1.6 Given $M \in \Re^{n \times n}$:

(i) (q, M) has finitely many solutions for all $q \in \Re^n$ if and only if M is nondegenerate;

(ii) if $q \in \Re^n$ is in the interior of a degenerate complementary cone then (q, M) has infinitely many solutions;

(iii) each degenerate complementary cone is associated with exactly one solution of (q, M) for each $q \in \Re^n$ that it contains (and, of course, it is associated with no solutions for the q it doesn't contain);

(iv) if $q \in \Re^n$ is nondegenerate then there is a bijective correspondence between solutions of (q, M) and complementary cones containing q.

□

The concept of complementary cones is first seen in Samelson, Thrall and Wesler (1958), and was later given a comprehensive treatment by Murty (1972). Proposition 1.6 is an expansion of theorems proved in Murty (1972). Before moving on to discuss other areas of LCP background material, it is important to bring up the following

DEFINITION 1.7 For $M \in \Re^{n \times n}$, we say the two complementary cones $\operatorname{pos} C(\alpha)$ and $\operatorname{pos} C(\beta)$, with $\alpha, \beta \in (\bar{n})$, are *adjacent* if and only if $|\alpha \vartriangle \beta| = 1$. That is, two distinct complementary cones are adjacent if they share a common face. If $\alpha \vartriangle \beta = \{i\}$, then that common face is $\operatorname{pos} C(\alpha)_{\cdot \bar{i}} = \operatorname{pos} C(\beta)_{\cdot \bar{i}}$.

DEFINITION 1.8 For $M \in \Re^{n \times n}$, we say the common face $\operatorname{pos} C_M(\alpha)_{\cdot \bar{i}}$ between the complementary cones $\operatorname{pos} C_M(\alpha)$ and $\operatorname{pos} C_M(\beta)$, where $\alpha \vartriangle \beta = \{i\}$, is *proper* if and only if $(\det C_M(\alpha))(\det C_M(\beta)) < 0$.

As $\det C_M(\alpha) = (-1)^{|\alpha|} \det M_{\alpha\alpha}$, we have

$$\operatorname{pos} C_M(\alpha)_{\cdot i} \text{ is proper if and only if } (\det M_{\alpha\alpha})(\det M_{\beta\beta}) > 0.$$

Geometrically, $\operatorname{pos} C_M(\alpha)_{\cdot i}$ is proper if and only if it is $(n-1)$-dimensional and the vectors $I_{\cdot i}$ and $-M_{\cdot i}$ lie on strictly opposite sides of $\operatorname{span} C_M(\alpha)_{\cdot i}$.

DEFINITION 1.9 For $M \in \Re^{n \times n}$, we say the common face $\operatorname{pos} C_M(\alpha)_{\cdot i}$ between the complementary cones $\operatorname{pos} C_M(\alpha)$ and $\operatorname{pos} C_M(\beta)$, where $\alpha \triangle \beta = \{i\}$, is *reflecting* if and only if $(\det C_M(\alpha))(\det C_M(\beta)) > 0$.

Similar to the above we have

$$\operatorname{pos} C_M(\alpha)_{\cdot i} \text{ is reflecting if and only if } (\det M_{\alpha\alpha})(\det M_{\beta\beta}) < 0.$$

Geometrically, $\operatorname{pos} C_M(\alpha)_{\cdot i}$ is reflecting if and only if it is $(n-1)$-dimensional and the vectors $I_{\cdot i}$ and $-M_{\cdot i}$ lie on the same side of $\operatorname{span} C_M(\alpha)_{\cdot i}$.

DEFINITION 1.10 For $M \in \Re^{n \times n}$, we say the common face $\operatorname{pos} C_M(\alpha)_{\cdot i}$ between the complementary cones $\operatorname{pos} C_M(\alpha)$ and $\operatorname{pos} C_M(\beta)$, where $\alpha \triangle \beta = \{i\}$, is *degenerate* if and only if $(\det C_M(\alpha))(\det C_M(\beta)) = 0$.

As above, it can be shown that $\operatorname{pos} C_M(\alpha)_{\cdot i}$ is degenerate if and only if $(\det M_{\alpha\alpha})(\det M_{\beta\beta}) = 0$, if and only if $\operatorname{pos} C_M(\alpha)_{\cdot i}$ is a face of a degenerate complementary cone.

For examples of the preceding definitions see to Figure 1.3, which shows $\mathrm{K}(M)$ for the matrix (1.13). Here $\operatorname{pos} C(\emptyset)_{\cdot 1}$ is proper, $\operatorname{pos} C(\overline{2})_{\cdot 2}$ is reflecting, and $\operatorname{pos} C(\overline{2})_{\cdot 1}$ and $\operatorname{pos} C(\emptyset)_{\cdot 2}$ are degenerate.

We now move on to consider the algebraic concept of principal transforms of the matrix M. For a more detailed discussion see Tucker (1960, 1963), Cottle and Dantzig (1968), and also Cottle (1974). Suppose we are given a matrix $M \in \Re^{m \times n}$ which is not necessarily square and can be permuted to

look like

$$M = \left[\begin{array}{c|c} M_{\alpha\beta} & M_{\alpha\hat{\beta}} \\ \hline M_{\hat{\alpha}\beta} & M_{\hat{\alpha}\hat{\beta}} \end{array}\right] \in \Re^{m\times n}.$$

Moreover, suppose $\alpha \in (\overline{m})$, $\beta \in (\overline{n})$, $|\alpha| = |\beta|$, and $\det M_{\alpha\beta} \neq 0$. We then say the matrix

$$\overline{M} = \left[\begin{array}{c|c} M_{\alpha\beta}^{-1} & -M_{\alpha\beta}^{-1}\, M_{\alpha\hat{\beta}} \\ \hline M_{\hat{\alpha}\beta}\, M_{\alpha\beta}^{-1} & M_{\hat{\alpha}\hat{\beta}} - M_{\hat{\alpha}\beta}\, M_{\alpha\beta}^{-1}\, M_{\alpha\hat{\beta}} \end{array}\right] \in \Re^{m\times n} \tag{1.14}$$

is a *pivotal transform* of M. We also say $\overline{M}$ is gotten from M by *block pivoting* on $M_{\alpha\beta}$. If $\alpha = \beta$, we then say $\overline{M}$ is a *principal* transform of M. Notice from (1.14) that if $\alpha \subseteq \gamma \in (\overline{n})$ then the principal transform of $M_{\gamma\gamma}$ resulting from a block pivot on $M_{\alpha\alpha}$ is just $(\overline{M})_{\gamma\gamma}$. In other words, the principal transform of a submatrix will be the submatrix of a principal transform. (The converse is not necessarily true.) The following two theorems are straightforward algebraic consequences of the definition of $\overline{M}$. They can be found, for example, in Cottle (1974) and Parsons (1970).

THEOREM 1.11 Given $M \in \Re^{m\times n}$ with $\overline{M} \in \Re^{m\times n}$ being the transform of M obtained by block pivoting on $M_{\alpha\beta}$, then for all $x \in \Re^{n}$ and $y \in \Re^{m}$ we have

$$\left[\begin{array}{c|c} M_{\alpha\beta} & M_{\alpha\hat{\beta}} \\ \hline M_{\hat{\alpha}\beta} & M_{\hat{\alpha}\hat{\beta}} \end{array}\right] \left[\begin{array}{c} x_{\beta} \\ \hline x_{\hat{\beta}} \end{array}\right] = \left[\begin{array}{c} y_{\alpha} \\ \hline y_{\hat{\alpha}} \end{array}\right]$$

if and only if

$$\left[\begin{array}{c|c} \overline{M}_{\alpha\beta} & \overline{M}_{\alpha\bar{\beta}} \\ \hline \overline{M}_{\bar{\alpha}\beta} & \overline{M}_{\bar{\alpha}\bar{\beta}} \end{array}\right] \left[\begin{array}{c} y_\alpha \\ \hline x_{\bar{\beta}} \end{array}\right] = \left[\begin{array}{c} x_\beta \\ \hline y_{\bar{\alpha}} \end{array}\right].$$

□

THEOREM 1.12 (Tucker (1960)) Given $M \in \Re^{n\times n}$ and $\alpha \in (\overline{n})$. If $\overline{M} \in \Re^{n\times n}$ is the principal transform of M obtained by block pivoting on $M_{\alpha\alpha}$, then for all $\beta \in (\overline{n})$

$$\det \overline{M}_{\beta\beta} = \frac{\det M_{\alpha\triangle\beta,\alpha\triangle\beta}}{\det M_{\alpha\alpha}}.$$

□

We will now obtain a few facts concerning principal transforms and their relation to the LCP.

THEOREM 1.13 Given $M \in \Re^{n\times n}$ and $q \in \Re^n$, consider the matrix $[\ M \mid q\ ] \in \Re^{n\times(n+1)}$ and let $[\ \overline{M} \mid \overline{q}\ ] \in \Re^{n\times(n+1)}$ be its principal transform obtained by blocking pivoting on $M_{\alpha\alpha}$ for some $\alpha \in (\overline{n})$. Then $|\mathrm{sol}(q, M)| = |\mathrm{sol}(\overline{q}, \overline{M})|$.

Proof. From Theorem 1.11 we have for any $w, z \in \Re^n$ that

$$\left[\begin{array}{c|c|c} M_{\alpha\alpha} & M_{\alpha\bar{\alpha}} & q_\alpha \\ \hline M_{\bar{\alpha}\alpha} & M_{\bar{\alpha}\bar{\alpha}} & q_{\bar{\alpha}} \end{array}\right] \left[\begin{array}{c} z_\alpha \\ \hline z_{\bar{\alpha}} \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} w_\alpha \\ \hline w_{\bar{\alpha}} \end{array}\right] \qquad (1.15)$$

if and only if

$$\left[\begin{array}{c|c|c} \overline{M}_{\alpha\alpha} & \overline{M}_{\alpha\bar\alpha} & \overline{q}_{\alpha} \\ \hline \overline{M}_{\bar\alpha\alpha} & \overline{M}_{\bar\alpha\bar\alpha} & \overline{q}_{\bar\alpha} \end{array}\right] \left[\begin{array}{c} w_{\alpha} \\ \hline z_{\bar\alpha} \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} z_{\alpha} \\ \hline w_{\bar\alpha} \end{array}\right]. \tag{1.16}$$

Hence, if $([w_\alpha, w_{\bar\alpha}], [z_\alpha, z_{\bar\alpha}]) \in \text{sol}(q, M)$ then $([z_\alpha, w_{\bar\alpha}], [w_\alpha, z_{\bar\alpha}]) \in \text{sol}(\overline{q}, \overline{M})$, and vice versa. This gives us a bijective correspondence between solutions to (q, M) and solutions to $(\overline{q}, \overline{M})$. Thus, the number of solutions must be the same for the two LCP's.

□

THEOREM 1.14 Given $M \in \Re^{n\times n}$, $q \in \Re^n$ and $\alpha \in (\overline{n})$, let $[\ \overline{M} \mid \overline{q}\ ] \in \Re^{n\times(n+1)}$ be the principal transform of $[\ M \mid q\ ] \in \Re^{n\times n}$ obtained by block pivoting on $M_{\alpha\alpha}$. Then $q \in \text{int K}(M)$ if and only if $\overline{q} \in \text{int K}(\overline{M})$.

Proof. For any $z, w, x \in \Re^n$, Theorem 1.11 implies that

$$\left[\begin{array}{c|c|c|c|c} M_{\alpha\alpha} & M_{\alpha\bar\alpha} & q_{\alpha} & I_{\alpha\alpha} & 0 \\ \hline M_{\bar\alpha\alpha} & M_{\bar\alpha\bar\alpha} & q_{\bar\alpha} & 0 & I_{\bar\alpha\bar\alpha} \end{array}\right] \left[\begin{array}{c} z_{\alpha} \\ \hline z_{\bar\alpha} \\ \hline 1 \\ \hline x_{\alpha} \\ \hline x_{\bar\alpha} \end{array}\right] = \left[\begin{array}{c} w_{\alpha} \\ \hline w_{\bar\alpha} \end{array}\right] \tag{1.17}$$

if and only if

$$\left[\begin{array}{cc|c|c|c}\overline{M}_{\alpha\alpha} & \overline{M}_{\alpha\bar\alpha} & \overline{q}_\alpha & -M_{\alpha\alpha}^{-1} & 0 \\ \hline \overline{M}_{\bar\alpha\alpha} & \overline{M}_{\bar\alpha\bar\alpha} & \overline{q}_{\bar\alpha} & 0 & I_{\bar\alpha\bar\alpha}\end{array}\right]\left[\begin{array}{c} w_\alpha \\ \hline z_{\bar\alpha} \\ \hline 1 \\ \hline x_\alpha \\ \hline x_{\bar\alpha}\end{array}\right] = \left[\begin{array}{c} z_\alpha \\ \hline w_{\bar\alpha}\end{array}\right]. \tag{1.18}$$

Notice that, as the columns of $M_{\alpha\alpha}^{-1}$ are linearly independent, the columns of

$$\left[\begin{array}{c|c} -M_{\alpha\alpha}^{-1} & 0 \\ \hline 0 & I_{\bar\alpha\bar\alpha}\end{array}\right] \tag{1.19}$$

span $\Re^n$. Suppose $q \in \mathrm{K}(M)$. We then have an $\epsilon > 0$ such that $x \in B(q, \epsilon)$ implies $\mathrm{sol}(q+x, M) \neq \emptyset$. Let $\overline{x} = (-M_{\alpha\alpha}^{-1} x_\alpha, x_{\bar\alpha})^T \in \Re^n$. Thus, by (1.17) and (1.18) we see that $\mathrm{sol}(q+x, M) \neq \emptyset$ implies $\mathrm{sol}(\overline{q}+\overline{x}, \overline{M}) \neq \emptyset$. As (1.19) spans $\Re^n$, the set of $\overline{x}$ corresponding to all $x \in B(q, \epsilon)$ contains an open ball around $\overline{q}$. Thus $\overline{q} \in \mathrm{int}\,\mathrm{K}(\overline{M})$. This proves one direction of the theorem. The other direction is proved by the same argument.

□

THEOREM 1.15 Given $M \in \Re^{n\times n}$, $q \in \Re^n$ and $\alpha \in (\overline{n})$, let $[\,\overline{M} \mid \overline{q}\,] \in \Re^{n\times(n+1)}$ be the principal transform of $[\,M \mid q\,] \in \Re^{n\times(n+1)}$ obtained by block pivoting on $M_{\alpha\alpha}$. Then $q \in \mathrm{int}\,\mathrm{pos}\,C_M(\beta)$ if and only if $\overline{q} \in \mathrm{int}\,\mathrm{pos}\,C_{\overline{M}}(\alpha \vartriangle \beta)$.

Proof. Suppose $q \in \mathrm{int}\,\mathrm{pos}\,C_M(\beta)$. Then there is an $\epsilon > 0$ such that $x \in B(q, \epsilon)$ implies $q + x \in \mathrm{pos}\,C_M(\beta)$, the latter thing implies there is a $(w, z) \in \mathrm{sol}(q + x, M)$ such that $w_\beta = 0$ and $z_{\bar\beta} = 0$. As before, let $\overline{x} = (-M_{\alpha\alpha}^{-1} x_\alpha, x_{\bar\alpha})^T \in \Re^n$. By (1.17) and (1.18) we see that $(\overline{w}, \overline{z}) = ([z_\alpha, w_{\bar\alpha}], [w_\alpha, z_{\bar\alpha}]) \in \mathrm{sol}(\overline{q}+\overline{x}, \overline{M})$. Hence, with $\gamma = \alpha \vartriangle \beta$, we have $\overline{w}_\gamma = 0$

and $\bar{z}_{\hat{\gamma}} = 0$. This means $\bar{q} + \bar{x} \in \operatorname{pos} C_{\overline{M}}(\gamma)$. Thus for the set of $\bar{x}$ corresponding to all $x \in B(q, \epsilon)$, which as before will contain an open ball around $\bar{q}$, we have $\bar{q} + \bar{x} \in \operatorname{pos} C_{\overline{M}}(\alpha \vartriangle \beta)$. Hence, $\bar{q} \in \operatorname{int} \operatorname{pos} C(\alpha \vartriangle \beta)$. The other direction of the theorem is proved by the same argument. □

The preceding theorems show that, from the standpoint of combinatorial topology, the structure of $K(M)$ is identical to the structure of $K(\overline{M})$. The positive orthant in $K(\overline{M})$ is identified with $\operatorname{pos} C(\alpha)$ in $K(M)$. Pivoting on $M_{\alpha\alpha}$ is, in essence, "swapping" the vectors $I_{\cdot\alpha}$ with the vectors $-M_{\cdot\alpha}$.

As our last topic, we turn to look at some classes of matrices that we will need. We will be discussing many more matrix classes in Chapter 5, but for now we will mention only the classes $\mathbf{P}$, $\mathbf{P}_0$, $\mathbf{Q}$, $\mathbf{Q}_0$, and $\mathbf{E}_0$.

We say a matrix $M \in \Re^{n \times n}$ is in $\mathbf{P}$ ($\mathbf{P}_0$) if and only if all its principal minors are positive (nonnegative). It is clear that membership in $\mathbf{P}$ or $\mathbf{P}_0$ is an *inherited* property, i.e., if a matrix is in $\mathbf{P}$ ($\mathbf{P}_0$) then all its principal submatrices are in $\mathbf{P}$ ($\mathbf{P}_0$). We also see, from Theorem 1.12, that if a matrix is in $\mathbf{P}$ ($\mathbf{P}_0$) then all its principal transforms are in $\mathbf{P}$ ($\mathbf{P}_0$). (This was first proved in Tucker (1963).) The main theorem concerning the geometric structure of P-matrices comes from Samelson, Thrall and Wesler (1958) and states

THEOREM 1.16 For $M \in \Re^{n \times n}$, $M \in \mathbf{P}$ if and only if $|\mathrm{sol}(q, M)| = 1$ for all $q \in \Re^n$. □

Another pair of matrix classes that are *defined* by the LCP are $\mathbf{Q}$ and $\mathbf{Q}_0$. A matrix $M \in \Re^{n \times n}$ is said to be in $\mathbf{Q}$ if and only if $\mathrm{sol}(q, M) \neq \emptyset$ for all $q \in \Re^n$, i.e., $K(M) = \Re^n$. It is clear that $\mathbf{P} \subseteq \mathbf{Q}$. However, the zero

matrix in any dimension is in $\mathbf{P}_0$ but not in $\mathbf{Q}$. Also, the matrix

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \tag{1.20}$$

is not in $\mathbf{P}_0$, so not in $\mathbf{P}$, but it is in $\mathbf{Q}$, as can be seen in Figure 1.4.

The definition of $\mathbf{Q}_0$ requires the concept of being "feasible" with respect to a LCP. We say $z \in \Re^n$ is *feasible* with respect to (q, M) if and only if z satisfies

$$z \geq 0, \tag{1.1}$$

$$Mz + q \geq 0. \tag{1.2}$$

We now can define $M \in \Re^{n \times n}$ to be in $\mathbf{Q}_0$ if and only if for all $q \in \Re^n$, for which there is a $z \in \Re$ which is feasible for (q, M), we have $\text{sol}(q, M) \neq \emptyset$. Clearly $\mathbf{Q} \subseteq \mathbf{Q}_0$. Notice that the negative of the identity matrix in any dimension is not in $\mathbf{P}_0$, but is in $\mathbf{Q}_0$ as then $(q, -I)$ has a feasible z if and only if $q \geq 0$, in which case $(q, 0) \in \text{sol}(q, M)$. Also the matrix

$$M = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \tag{1.21}$$

is in $\mathbf{P}_0$ but not in $\mathbf{Q}_0$, for when $q = (-1, 0)^T$ we find $\text{sol}(q, M) \neq \emptyset$ although $z = (0, 1)^T$ is feasible. From Eaves (1971), we have the following geometric result pertaining to $\mathbf{Q}_0$-matrices.

THEOREM 1.17 Given $M \in \Re^{n \times n}$, $M \in \mathbf{Q}_0$ if and only if $\text{K}(M)$ is a convex set in $\Re^n$.

□

Before leaving this section, we mention the matrix class $\mathbf{E}_0$. A matrix $M \in \Re^{n \times n}$ is said to be *semi-monotone*, denoted $M \in \mathbf{E}_0$, if and only if

for all $x \in \Re^n$, where $0 \neq x \geq 0$, there is an index $k \in \overline{n}$ for which $x_k > 0$ and $(Mx)_k \geq 0$. (This class was introduced in Eaves (1969).) Consider the matrices

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}.$$

(1.22) (1.23)

Notice that the matrix (1.22) is in $\mathbf{E_0}$ but not in $\mathbf{P_0}$. It isn't in $\mathbf{Q_0}$ since $z = (1,0)^T$ is feasible for $q = (1,-1)^T$, yet there is no solution to the LCP with this q and matrix (1.22). Also, matrix (1.23) is in $\mathbf{Q}$, as can be seen in Figure 1.5, but is not in $\mathbf{E_0}$. It is fairly obvious that

$$M > 0 \qquad \Rightarrow \qquad M \in \mathbf{E_0}\,. \tag{1.24}$$

It is also fairly obvious that being in $\mathbf{E_0}$ is an *inherited* property, i.e., if a matrix is in $\mathbf{E_0}$ then so are all its principal submatrices. For if the vector $x \in \Re^n$ with $0 \neq x_\alpha \geq 0$ is such that $(M_{\alpha\alpha} x_\alpha)_k < 0$ for all $k \in \alpha$ where $x_k > 0$, then letting $x_{\bar{\alpha}} = 0$ we have $0 \neq x \geq 0$ with $(Mx)_k < 0$ for all $k \in (\overline{n})$ where $x_k > 0$. A is less obvious fact is the following (see Fiedler and Pták (1966), Lemke (1970) and Eaves (1971)).

THEOREM 1.18 $\mathbf{P_0} \subseteq \mathbf{E_0}$. □

1.3 Notation

For easy reference, this section lists the notation that will be used in this work and specifically the notation which is not standard.

Item	Explanation
$\bar{n}$	The set $\{1,2,3,\ldots,n\}$.
α, β, γ, etc.	Index sets. Example: the ordered k-tuple $\alpha = (\alpha_1,\ldots,\alpha_k)$ with $1 \le \alpha_1 < \cdots < \alpha_k \le n$.
$(\bar{n})$	The collection of all index sets formed from $\bar{n}$ (including the empty set, $\emptyset$).
$\hat{\alpha}$	The index set "complementary" to α (relative to $\bar{n}$). $\hat{\alpha}$ is obtained from $(1,2,\ldots,n)$ by deleting the components in α.
$\hat{\imath}$	$\hat{\alpha}$ for $\alpha = \{i\}$.
$\Re^{m\times n}$	The class of all real $m \times n$ matrices.
$\mathbf{Z}_+$	The class of all positive integers.
$M_{\alpha\beta}$	The submatrix of M with rows indexed by α and columns indexed by β. If $\alpha = \beta$ we say then call $M_{\alpha\alpha}$ a *principal submatrix* of M. The determinant of a principal submatrix of M is called a *principal minor* of M. By convention $\det M_{\emptyset\emptyset} = 1$.
$M_{\alpha\beta}^{-1}$	$(M_{\alpha\beta})^{-1}$.
$M_{i\cdot}$	The i^{th} row of M.
$M_{\cdot j}$	The j^{th} column of M.
$M_{\alpha\cdot}$	The rows of M indexed by α.
$M_{\cdot\beta}$	The columns of M indexed by β.

z_α	The entries of the vector z indexed by α.
$C_M(\alpha)$ or $C(\alpha)$	A *complementary matrix* relative to M and the index set α. If $C = C(\alpha)$, then $$C_{\cdot j} = \begin{cases} -M_{\cdot j} & \text{if } j \in \alpha \\ I_{\cdot j} & \text{if } j \notin \alpha \end{cases}$$ The subscript M is normally dropped when it is clear from the context.
$\operatorname{span} C$	The column space of the matrix C.
$\operatorname{aff} X$	The *affine hull* of the set X. This is the set $\{ x + \theta(y - x) : x, y \in X, \text{ and } \theta \in \Re \}$.
$\dim X$	The dimension of the affine hull of the set X. This is the minimum number of columns needed in a matrix C so that, given some $q \in X$, we have $\operatorname{aff} X = \{ q + z : z \in \operatorname{span} C \}$.
$\operatorname{pos} C$	The set $\{ Cx : x \geq 0 \}$ where C is a matrix (not necessarily square). If C is a complementary matrix relative to M and some $\alpha \in (\bar{n})$, then $\operatorname{pos} C$ is called a *complementary cone.* A complementary cone is said to be *full* or *nondegenerate* if $\det C \neq 0$. Otherwise, it is called *degenerate.*
$\mathrm{K}(M)$	The set $$\bigcup_{\alpha \in (\bar{n})} \operatorname{pos} C_M(\alpha).$$

$K(M)$	The set $$\bigcup_{\substack{\alpha \in (\overline{n}) \\ i \in \overline{n}}} \operatorname{pos} C_M(\alpha).$$
$B(q,\epsilon)$	The open ball centered at q with radius ϵ. This is the set $\{x \in \Re^n : \|x - q\| < \epsilon\}$.
$\operatorname{int} X$	The *relative interior* of the set X with respect to $\operatorname{aff} X$. This is the set of all $q \in X$ such that there is some $\epsilon > 0$ such that $\operatorname{aff} X \cap B(q,\epsilon) \subseteq X$.
∂X	The *relative boundary* of the set X with respect to $\operatorname{aff} X$. Thus $\partial X = X \setminus \operatorname{int} X$.
$\overline{X}$	The *closure* of the set X. This is the set of all $z \in \Re^n$ such that for all $\epsilon > 0$ there is a $q \in X$ where $q \in B(z,\epsilon)$.
P	$\cup_n \{ M \in \Re^{n \times n} : \det M_{\alpha\alpha} > 0, \text{ for all } \alpha \in (\overline{n}) \}$.
$\mathbf{P_0}$	$\cup_n \{ M \in \Re^{n \times n} : \det M_{\alpha\alpha} \geq 0, \text{ for all } \alpha \in (\overline{n}) \}$.
Q	$\cup_n \{ M \in \Re^{n \times n} : \mathrm{K}(M) = \Re^n \}$.
$\mathbf{Q_0}$	$\cup_n \{ M \in \Re^{n \times n} : \mathrm{K}(M) = \operatorname{pos}[\, I \mid -M \,] \}$.
$\mathbf{E_0}$	$\cup_n \{ M \in \Re^{n \times n} : 0 \neq x \geq 0 \Rightarrow \exists_k \;\; x_k > 0 \;\&\; (Mx)_k \geq 0 \}$. Matrices in this class are said to be *semi-monotone.*
$\|X\|$	The cardinality of the set X.
(q, M)	The **LCP** given by (1.1), (1.2) and (1.3).

sol(q, M)	The set of all solutions of the **LCP** (q, M).
$\alpha \triangle \beta$	The *symmetric difference* of α and β. This is the set $(\alpha \setminus \beta) \cup (\beta \setminus \alpha)$.
supp x	The *support* of the vector x. This is the set $\{j : x_j \neq 0\}$.

One last point before ending this list: we say a set X is *star-shaped* at q if and only if for every $z \in X$ we have

$$\{\lambda q + (1 - \lambda) z : 0 \leq \lambda \leq 1\} \subseteq X .$$

This says that the line segment between q and z is contained in X.

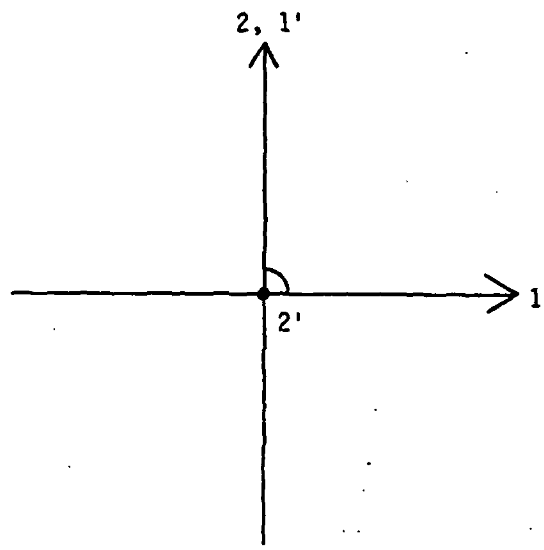

Figure 1.1

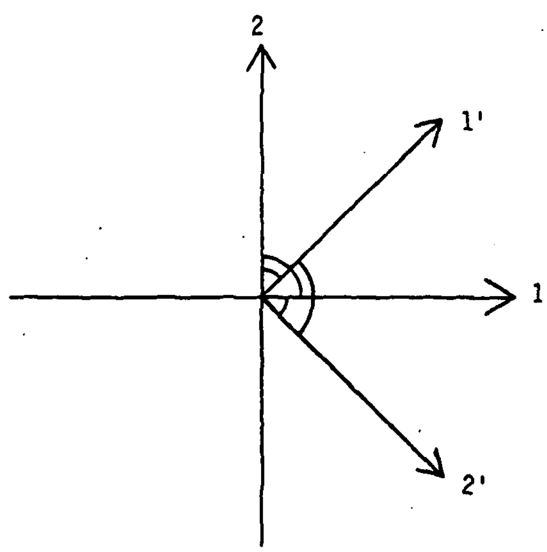

Figure 1.2

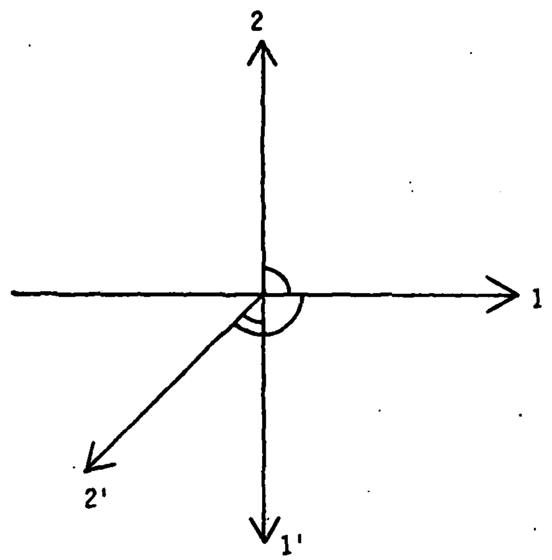

Figure 1.3

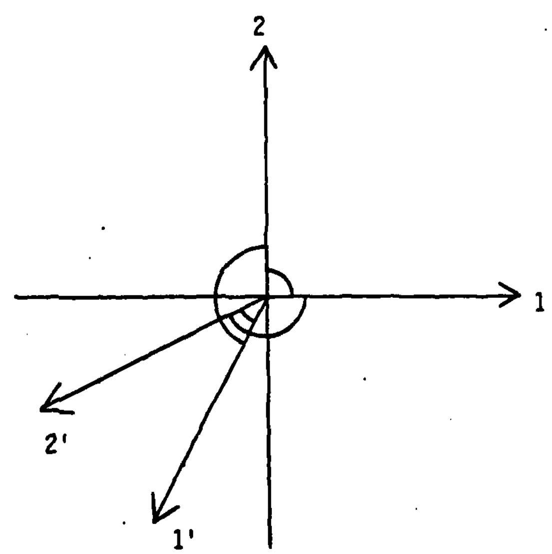

Figure 1.4

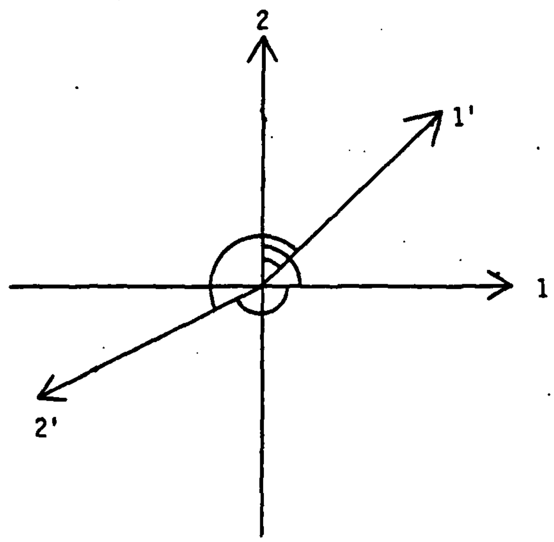

Figure 1.5

CHAPTER 2.
THE CLASS OF U-MATRICES

2.1 Preliminary Definitions and Results

In Chapter 1 we exhibited several matrix classes that are related to the LCP. It is often the case that one useful class of matrices leads to the consideration of other interesting matrix classes, gotten by weakening or strengthening the conditions that define the original class. For example, the class **P** suggests considering the more general class $\mathbf{P}_0$. The class **P**, viewed as the class of all matrices M for which (q, M) has a unique solution for all q, suggests defining the class **Q** by dropping the uniqueness requirement and just requiring that for each q a solution to (q, M) must exist. The class **Q**, in turn, gives rise to the class $\mathbf{Q}_0$ when we relax the definition to require only that (q, M) have a solution whenever (1.1) and (1.2) alone are satisfiable.

We presently wish to understand the basic geometric structure which gives rise to unique solutions to (q, M). With this in mind, we consider the following class of matrices

$$\bigcup_n \{\, M \in \Re^{n\times n} : |\mathrm{sol}(q, M)| = 1, \text{ for all } q \in \mathrm{K}(M) \,\}\,.$$

This matrix class is obtained from **P** by relaxing the requirement that (q, M) have a solution for all q, as **Q** was obtained by dropping the uniqueness requirement. However, as we will see later, this "new" matrix class consists of nothing but **P**. While this is an insightful result by itself, a subtler weakening of the definition of **P**-matrices is needed to get an appropriate matrix class for our analysis. We find that the appropriate class to study is embodied in the following definition.

DEFINITION 2.1. A matrix A will be said to be a U-matrix, $A \in \mathbf{U}$, if and only if

$$A \in \bigcup_n \{\, M \in \Re^{n\times n} : |\mathrm{sol}(q, M)| = 1, \text{ for all } q \in \mathrm{int}\ \mathrm{K}(M) \,\}\,.$$

The goal of the next section will be to develop a characterization for the class **U**. Before embarking on this task, we give two examples which verify that **U** consists of more than just **P**; we also discuss some needed definitions and results.

EXAMPLE 2.2 Let

$$M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

In this case,

$$\mathrm{K}(M) = \{\, q \in \Re^2 : q_1 + q_2 \geq 0 \,\}$$

as shown in Figure 2.1. Note here that (q, M) has a unique solution for all q satisfying $q_1 + q_2 > 0$ including those of the form

$$\begin{bmatrix} q_1 \\ 0 \end{bmatrix} \quad q_1 > 0 \qquad \text{and} \qquad \begin{bmatrix} 0 \\ q_2 \end{bmatrix} \quad q_2 > 0$$

for which the solution to (q, M) is degenerate.

EXAMPLE 2.3 Let

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In this case,

$$\mathrm{K}(M) = \Re^2_+ \cup \Re^2_-$$

as shown in Figure 2.2. Here the problem has a unique solution for all q satisfying $q > 0$ or $q < 0$.

Notice that $\mathrm{K}(M)$ is convex in Example 2.2 and nonconvex in Example 2.3. In both cases, $|\mathrm{sol}(q, M)| = \infty$ for all $q \in \partial \mathrm{K}(M)$.

Perhaps the most important fact underlying the study of uniqueness is expressed by

LEMMA 2.4 If $M \in \Re^{n \times n}$, the following are equivalent:

(i) $M \in \mathbf{E}_0$ (that is, M is semi-monotone);

(ii) (q, M) has a unique solution for all $q > 0$;

(iii) for all $\alpha \in (\overline{n})$, the system

$$M_{\alpha\alpha} x_\alpha < 0, \qquad x_\alpha \geq 0$$

has no solution.

□

The equivalence of (i) and (ii) was shown by Eaves (1971). The equivalence of (i) and (iii) was shown by Lemke (1970).

Since $\mathrm{int}\, \Re^n_+ \subseteq \mathrm{int}\, \mathrm{K}(M)$ for *any* $M \in \Re^{n \times n}$, it follows immediately

from the definitions that

$$\mathbf{U} \subseteq \mathbf{E}_0 \,. \tag{2.1}$$

However, the inclusion is proper as shown by

$$M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \in \mathbf{E}_0$$

for which $\mathrm{K}(M) = \Re^2$. In this instance $M \in \mathbf{E}_0$ as $M > 0$. $M \in \mathbf{Q}$, as seen by Figure 2.3, so every point $q \in \Re^2$ is interior to $\mathrm{K}(M)$. But some problems (q, M) do not have unique solutions, for otherwise M would belong to $\mathbf{P}$ which it does not.

Let $M \in \Re^{n\times n}$ and $q \in \Re^n$ be given. If the matrix $[\ \overline{M} \mid \overline{q}\ ]$ is a principal transform of $[\ M \mid q\ ]$ then, by Theorems 1.14 and 1.13, respectively, we know that $q \in \mathrm{int}\, \mathrm{K}(M)$ if and only if $\overline{q} \in \mathrm{int}\, \mathrm{K}(\overline{M})$, and that $|\mathrm{sol}(q, M)| = |\mathrm{sol}(\overline{q}, \overline{M})|$. From this we find

$$M \in \mathbf{U} \quad \Leftrightarrow \quad \overline{M} \in \mathbf{U}.$$

This leads us to the following definition.

DEFINITION 2.5 If $M \in \Re^{n\times n}$, we say M is *fully semi-monotone* if and only if every principal transform of M is semi-monotone. We denote the class of such matrices by $\mathbf{E}_0^{\mathbf{f}}$.

We remark that $\mathbf{E}_0^{\mathbf{f}} \subseteq \mathbf{E}_0$ as the "empty pivot" is always legitimate: M is always a principal transform of itself. Notice that being in $\mathbf{E}_0^{\mathbf{f}}$ is an inherited property of matrices. For, from Chapter 1, we know that being in $\mathbf{E}_0$ is an inherited property, and also that a principal transform of a principal submatrix will be a principal submatrix of a principal transform.

The matrices used in Examples 2.2 and 2.3 show that $\mathbf{E}_0^{\mathbf{f}}$ is a nonempty class. As a matter of fact, $\mathbf{E}_0^{\mathbf{f}}$ contains $\mathbf{P}_0$. This follows as any principal transform of a $\mathbf{P}_0$-matrix belongs to $\mathbf{P}_0$, and as $\mathbf{P}_0 \subseteq \mathbf{E}_0$. The matrix used in Example 2.3 shows that $\mathbf{P}_0 \subseteq \mathbf{E}_0^{\mathbf{f}}$ is a proper inclusion.

Our remarks above the definition imply that

$$\mathbf{U} \subseteq \mathbf{E}_0^{\mathbf{f}} \tag{2.2}$$

which strengthens (2.1). But, again, the inclusion is proper. Indeed,

$$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathbf{E}_0^{\mathbf{f}} ,$$

but with $q = (1, 0)$ the problem (q, M) has the solutions

$$\begin{aligned} (w^1, z^1) &= (1, 0, 0, 0) \\ (w^2, z^2) &= (0, 0, 0, 1) . \end{aligned}$$

The corresponding cone $\mathrm{K}(M)$, shown in Figure 2.3, is quite revealing. Notice that $\operatorname{int} \mathrm{K}(M)$ contains the interior of the degenerate complementary cone $\operatorname{pos} C(\{2\})$.

2.2 Characterization of U-matrices

We have seen in the last section that $\mathbf{U} \subset \mathbf{E}_0^{\mathbf{f}}$. The task now is to find precise conditions under which a matrix in $\mathbf{E}_0^{\mathbf{f}}$ will also be in $\mathbf{U}$. It turns out to be easier to state exact conditions for when an $\mathbf{E}_0^{\mathbf{f}}$-matrix is *not* in $\mathbf{U}$. The main result of this section is:

THEOREM 2.6 Let $M \in \Re^{n \times n}$. Then $M \notin \mathbf{U}$ if and only if either $M \notin \mathbf{E}_0^f$ or there exist $\alpha, \beta \in (\overline{n})$ and $i, j \in \overline{n}$ such that

(i) $\alpha \neq \beta$, $i \neq j$,

(ii) $(\det M_{\alpha\alpha})(\det M_{\beta\beta}) \neq 0$ and there exists a nonzero vector $v \in \Re^n$ such that $v^T C(\alpha)_{\cdot \hat{\imath}} = v^T C(\beta)_{\cdot \hat{\jmath}} = 0$,

(iii) there exists $x \in \Re^{n-1}$ with $x > 0$ and $C(\alpha)_{\cdot \hat{\imath}}\, x \in \operatorname{pos} C(\beta)_{\cdot \hat{\jmath}}$.

Taken together, conditions (i), (ii), and (iii) say that there are two full complementary cones which have an $(n-1)$-dimensional intersection on two differently-labelled faces.

To prove Theorem 2.6, we first prove two lemmas.

LEMMA 2.7 Let $M \in \Re^{n \times n}$. $M \in \mathbf{E}_0^f$ if and only if for all $\alpha, \beta \in (\overline{n})$ with $\det M_{\alpha\alpha} \neq 0$ and $\alpha \neq \beta$ we have

$$\operatorname{int} \operatorname{pos} C(\alpha) \cap \operatorname{pos} C(\beta) = \emptyset .$$

Proof. Let $[\,\overline{M} \mid \overline{q}\,]$ be the principal transform of $[\,M \mid q\,]$ gotten by block pivoting on $M_{\alpha\alpha}$. We know, by Proposition 1.15, that $q \in \operatorname{int} \operatorname{pos} C_M(\alpha)$ if and only if $\overline{q} \in \operatorname{int} \operatorname{pos} C_{\overline{M}}(\emptyset)$ [if and only if $\overline{q} > 0$]. If we assume that $M \in \mathbf{E}_0^f$, then $\overline{M} \in \mathbf{E}_0$. Letting $C = C_M$ and using Proposition 1.13 with Lemma 2.4 we conclude that

$$q \in \operatorname{int} \operatorname{pos} C(\alpha) \quad \Rightarrow \quad |\operatorname{sol}(q, M)| = 1 . \tag{2.3}$$

For $q \in \operatorname{int} \operatorname{pos} C(\alpha)$, we have $C(\alpha)^{-1} q = x > 0$ giving the solution $(w, z) \in \operatorname{sol}(q, M)$, where $z_\alpha = x_\alpha > 0$ and $w_{\bar{\alpha}} = x_{\bar{\alpha}} > 0$. If $q \in \operatorname{pos} C(\beta)$,

then there is a solution $(\tilde{w}, \tilde{z}) \in \text{sol}(q, M)$ with $\tilde{w}_\beta = 0$, and as $\alpha \neq \beta$, we have $(w, z) \neq (\tilde{w}, \tilde{z})$ contradicting (2.3).

Conversely, if we knew that $\text{int pos}\, C(\alpha)$ intersected no other complementary cones, then, as above, $x = C(\alpha)^{-1} q$ would give us a solution to (q, M), and it would be the only solution. Thus (2.3) is valid; again by Proposition 1.13 and Lemma 2.4, we have $\overline{M} \in \mathbf{E}_0$. Since this holds for all $\alpha \in (\overline{n})$ for which $\det M_{\alpha\alpha} \neq 0$, we have $M \in \mathbf{E}_0^{\mathrm{f}}$.

□

The preceding lemma says that when $M \in \mathbf{E}_0^{\mathrm{f}}$, no point in the interior of a full complementary cone lies in any other complementary cone.

LEMMA 2.8 Let $\mathcal{L}$ be an n-dimensional linear subspace of $\Re^p$, and let A and B belong to $\Re^{p \times m}$ where $m \geq n$. If, for $i, j \in \overline{m}$,

(i) $\text{span}\, A = \text{span}\, B = \mathcal{L}$,

(ii) $\text{int pos}\, A_{\cdot\bar{i}} \cap \text{int pos}\, B_{\cdot\bar{j}} \neq \emptyset$,

(iii) $\text{span}\, A_{\cdot\bar{i}} = \text{span}\, B_{\cdot\bar{j}} \neq \mathcal{L}$,

(iv) $A_{\cdot i}$ and $B_{\cdot j}$ lie on the same side of $\text{span}\, B_{\cdot\bar{j}}$ (relative to $\mathcal{L}$),

then

$$\text{int pos}\, A \cap \text{int pos}\, B \neq \emptyset.$$

Proof. From (ii), we have the existence of a positive vector $x \in \Re^{2m-2}$ such that

$$[A_{\cdot\bar{i}}, -B_{\cdot\bar{j}}]\, x = 0, \quad x > 0. \tag{2.4}$$

If the conclusion were false, there would be no vector $\overline{x}$ such that

$$[A, -B]\, \overline{x} = 0, \quad \overline{x} > 0.$$

Then, by Stiemke's alternative theorem (see Dantzig (1963)) there would exist a vector $\overline{u}$ such that

$$0 \neq \overline{u}^T[A, -B] \geq 0.$$

(Without loss of generality, we may assume $\overline{u} \in \mathcal{L}$.) But, by the same alternative theorem, the existence of a solution to (2.4) implies the *nonexistence* of a solution to

$$0 \neq u^T[A_{\cdot\bar{\imath}}, -B_{\cdot\bar{\jmath}}] \geq 0.$$

From this we deduce that

$$\overline{u}^T A_{\cdot\bar{\imath}} = \overline{u}^T B_{\cdot\bar{\jmath}} = 0.$$

Thus $\overline{u}$ is orthogonal to the span of $A_{\cdot\bar{\imath}}$ (which equals the span of $B_{\cdot\bar{\jmath}}$). Yet $\overline{u}^T A_{\cdot i} \geq 0 \geq \overline{u}^T B_{\cdot j}$. Thus $A_{\cdot i}$ and $B_{\cdot j}$ lie on opposite sides of $\operatorname{span} B_{\cdot\bar{\jmath}}$, since by (iii) neither can lie in $\operatorname{span} B_{\cdot\bar{\jmath}}$, a contradiction.

□

We remark that this lemma could be made stronger; e.g., we could allow $A \in \Re^{p \times r}$, $B \in \Re^{p \times s}$, $i \in \bar{r}$, $j \in \bar{s}$ and replace (i) and (iii) with

(i′) $\operatorname{span} A \subseteq \operatorname{span} B = \mathcal{L}$

(iii′) $\operatorname{span} A \neq \operatorname{span} A_{\cdot\bar{\imath}} \subseteq \operatorname{span} B_{\cdot\bar{\jmath}} \neq \mathcal{L}$.

However, stronger results are not needed in what follows.

Proof of Theorem 2.6: Sufficiency. As we have already noted, $\mathbf{U} \subset \mathbf{E}_0^{\mathbf{f}}$, so $M \notin \mathbf{E}_0^{\mathbf{f}}$ implies $M \notin \mathbf{U}$. Suppose then that $M \in \mathbf{E}_0^{\mathbf{f}}$ and the three conditions of Theorem 2.6 hold. Let α, β, i, j, v, and x be as described therein. Define

$$H = \{q : v^T q = 0\}.$$

Then by (ii)

$$\operatorname{pos} C(\alpha)_{\cdot\bar{\imath}} \cup \operatorname{pos} C(\beta)_{\cdot\bar{\jmath}} \subseteq H .$$

Clearly H is $(n-1)$-dimensional. By (ii), $\operatorname{pos} C(\alpha)_{\cdot\bar{\imath}}$ and $\operatorname{pos} C(\beta)_{\cdot\bar{\jmath}}$ are also $(n-1)$-dimensional. Condition (iii) implies

$$\operatorname{int} \operatorname{pos} C(\alpha)_{\cdot\bar{\imath}} \cap \operatorname{pos} C(\beta)_{\cdot\bar{\jmath}} \neq \emptyset .$$

In fact, the stronger assertion

$$\operatorname{int} \operatorname{pos} C(\alpha)_{\cdot\bar{\imath}} \cap \operatorname{int} \operatorname{pos} C(\beta)_{\cdot\bar{\jmath}} \neq \emptyset \tag{2.5}$$

must also hold. To see this, let $q = C(\alpha)_{\cdot\bar{\imath}}\, x$. As q is interior to $\operatorname{pos} C(\alpha)_{\cdot\bar{\imath}}$, the dimension statements above imply that for some $\epsilon > 0$, *all* points in H within a distance ϵ from q belong to $\operatorname{int} \operatorname{pos} C(\alpha)_{\cdot\bar{\imath}}$. But clearly $\operatorname{pos} C(\beta)_{\cdot\bar{\jmath}}$, which lies in H, contains interior points within ϵ of q ; hence (2.5) is valid.

Certainly $C(\alpha)_{\cdot i}$ and $C(\beta)_{\cdot j}$ do not lie in H. If they lie on the same side of H, then as $\operatorname{pos} C(\alpha)$ and $\operatorname{pos} C(\beta)$ are full cones, Lemma 2.8 implies that $\operatorname{int} \operatorname{pos} C(\alpha) \cap \operatorname{int} \operatorname{pos} C(\beta) \neq \emptyset$, contradicting Lemma 2.7. So $C(\alpha)_{\cdot i}$ and $C(\beta)_{\cdot j}$ lie on opposite sides of H. Hence

$$\operatorname{int} \operatorname{pos} C(\alpha)_{\cdot\bar{\imath}} \cap \operatorname{int} \operatorname{pos} C(\beta)_{\cdot\bar{\jmath}} \subseteq \operatorname{int}\{ \operatorname{pos} C(\alpha) \cup \operatorname{pos} C(\beta) \} \subseteq \operatorname{int} \mathrm{K}(M) .$$

Let $\gamma = \alpha \triangle \{ i \}$. Then

$$C(\gamma)_{\cdot\bar{\imath}} = C(\alpha)_{\cdot\bar{\imath}} \qquad \text{and} \qquad C(\gamma)_{\cdot i} = C(\beta)_{\cdot i} .$$

Since $i \neq j$, we have $\operatorname{pos} C(\gamma) \subseteq \operatorname{span} C(\alpha)_{\cdot\bar{\imath}}$; this implies $\operatorname{pos} C(\gamma)$ is a degenerate cone. However,

$$\operatorname{int} \operatorname{pos} C(\alpha)_{\cdot\bar{\imath}} \cap \operatorname{int} \operatorname{pos} C(\beta)_{\cdot\bar{\jmath}} \subseteq \operatorname{pos} C(\gamma) .$$

As this intersection is nonempty, there exist points of $\operatorname{pos} C(\gamma)$ in $\operatorname{int} \mathrm{K}(M)$, hence there are points $\tilde{q} \in \operatorname{int} \operatorname{pos} C(\gamma) \cap \operatorname{int} \mathrm{K}(M)$, and so $(\tilde{q}, M)$ will have more than one solution. That is, $M \notin \mathbf{U}$.

Necessity. We assume $M \notin \mathbf{U}$. Then $|\operatorname{sol}(q, M)| > 1$ for some q belonging to $\operatorname{int} \mathrm{K}(M)$. Considering what must be proved, we assume $M \in \mathbf{E}_0^{\mathbf{f}}$ and show that the three conditions are satisfied. There are two cases.

Case 1: q is in the intersection of two full complementary cones. Assume for the moment that one of the cones is $\operatorname{pos} C(\emptyset)$, i.e., the nonnegative orthant. Let $\operatorname{pos} C(\mu)$ be the other cone where $\mu \neq \emptyset$. Then there exists a unique vector $x \geq 0$ such that

$$C(\mu)\, x = q \geq 0 .$$

If $x_\mu = 0$, i.e., the solution does not use any columns from $-M$ but only columns from I, then by the uniqueness of x, we have $x = q$, and the solutions that arise from $C(\emptyset)$ and $C(\mu)$ are the same. If $x_\mu \neq 0$, we may assume $x_\mu > 0$. (If it is not, we may replace μ by $\sigma = \operatorname{supp} x$. Then $C(\sigma)\, x = C(\mu)\, x$. If $\operatorname{pos} C(\sigma)$ is degenerate, the argument of Case 2 applies.) Thus, as $C(\mu) = q$, we have

$$-M_{\mu\mu} x_\mu = q_\mu \geq 0, \qquad x_\mu > 0 .$$

But $\det M_{\mu\mu} \neq 0$, and $M_{\mu\mu} \in \mathbf{E}_0^{\mathbf{f}}$ as $M \in \mathbf{E}_0^{\mathbf{f}}$. Therefore having $-M_{\mu\mu} x_\mu \geq 0$ with $x_\mu > 0$ says, with respect to the **LCP** $(q_\mu, M_{\mu\mu})$, that an interior point of a full complementary cone is contained in $\Re_+^{|\mu|}$, another complementary cone. This contradicts Lemma 2.7.

For two full complementary cones, say $\operatorname{pos} C(\lambda)$ and $\operatorname{pos} C(\mu)$, the argument just given can be made to apply by performing a principal pivot

on $M_{\lambda\lambda}$. (Let the resulting matrix be $\overline{M}$ and use the cones $\operatorname{pos} C_{\overline{M}}(\emptyset)$, $\operatorname{pos} C_{\overline{M}}(\lambda \triangle \mu)$, and the correspondence between the cone structures of $\mathrm{K}(M)$ and $\mathrm{K}(\overline{M})$.) Either way, Case 1 cannot occur.

Case 2: q belongs to a degenerate cone. We now assume $\det M_{\mu\mu} = 0$ and

$$q \in \operatorname{pos} C(\mu) \cap \operatorname{int} \mathrm{K}(M) . \tag{2.6}$$

Let

$$\dim \operatorname{pos} C(\mu) = s, \qquad 0 < s < n.$$

Note that if $s = 0$, then $C(\mu) = -M = 0$. But then M belongs to U.

From (2.6) we have

$$\dim[\operatorname{pos} C(\mu) \cap \operatorname{int} \mathrm{K}(M)] = s .$$

Thus

$$\dim \left\{ \bigcup_{\lambda} [\operatorname{pos} C(\mu) \cap \operatorname{int} \mathrm{K}(M) \cap \operatorname{pos} C(\lambda)] : \det C(\lambda) \neq 0 \right\} \quad = \quad s$$

as $\operatorname{int} \mathrm{K}(M)$ is contained in the union of the full complementary cones. Since the union is finite, there exists a $\beta \in (\overline{n})$ with $\det C(\beta) \neq 0$ and

$$\dim[\operatorname{pos} C(\mu) \cap \operatorname{int} \mathrm{K}(M) \cap \operatorname{pos} C(\beta)] = s .$$

Lemma 2.7 says $\operatorname{pos} C(\mu) \cap \operatorname{int} \operatorname{pos} C(\beta) \neq \emptyset$, so

$$\operatorname{pos} C(\mu) \cap \operatorname{int} \mathrm{K}(M) \cap \operatorname{pos} C(\beta) \subseteq \partial \operatorname{pos} C(\beta) .$$

Since $\operatorname{pos} C(\mu) \cap \operatorname{pos} C(\beta)$ is a convex cone and $\operatorname{int} \operatorname{pos} C(\beta) \subseteq \operatorname{int} \mathrm{K}(M)$, it follows that

$$\operatorname{pos} C(\mu) \cap \operatorname{int} \mathrm{K}(M) \cap \operatorname{pos} C(\beta) \subseteq \operatorname{pos} C(\beta)_{\cdot \jmath}$$

for some $\jmath$. As

$$\dim \operatorname{pos} C(\mu) = s = \dim[\operatorname{pos} C(\mu) \cap \operatorname{pos} C(\beta)_{\cdot \jmath}],$$

we have $\operatorname{pos} C(\mu) \subseteq \operatorname{span} C(\beta)_{\cdot \jmath}$. The $(n-1)$-dimensional subspace

$$H = \operatorname{span} C(\beta)_{\cdot \jmath}$$

is the common boundary of the two closed half-spaces H^+ and H^-. Let H^+ contain $C(\beta)_{\cdot j}$. Now

$$\operatorname{int} \mathrm{K}(M) \cap \operatorname{pos} C(\beta)_{\cdot \jmath} \neq \emptyset,$$

whence

$$\dim[\operatorname{int} \mathrm{K}(M) \cap \operatorname{pos} C(\beta)_{\cdot \jmath}] = n - 1.$$

Suppose

$$\left.\begin{array}{l} \det C(\lambda) \neq 0 \\ \operatorname{pos} C(\lambda) \cap \operatorname{int} H^- \neq \emptyset \end{array}\right\} \Rightarrow \dim[\operatorname{pos} C(\lambda) \cap \operatorname{int} \mathrm{K}(M) \cap \operatorname{pos} C(\beta)_{\cdot \jmath}] < n-1.$$

Then there exists $q \in \operatorname{int} \mathrm{K}(M) \cap \operatorname{pos} C(\beta)_{\cdot \jmath}$ contained in no full cone that intersects H^-. Thus, there exists a number $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0]$

$$B(\epsilon, q) \cap \operatorname{int} H^- \cap \operatorname{int} \mathrm{K}(M) = \emptyset$$

since $\operatorname{int} \mathrm{K}(M)$ is in the union of the full complementary cones. But as $q \in H$, we have $B(\epsilon, q) \cap \operatorname{int} H^{-} \neq \emptyset$. Thus $\operatorname{int} \mathrm{K}(M)$ does not contain an open ball around $q \in \operatorname{int} \mathrm{K}(M)$, a contradiction. This implies there exists $\alpha \in (\bar{n})$ with $\det C(\alpha) \neq 0$, $\operatorname{pos} C(\alpha) \cap \operatorname{int} H^{-} \neq \emptyset$, and

$$\dim[\operatorname{pos} C(\alpha) \cap \operatorname{int} \mathrm{K}(M) \cap \operatorname{pos} C(\beta)_{\cdot \bar{j}}] = n - 1 .$$

Since $\operatorname{pos} C(\beta) \subseteq H^{+}$, it is clear that $\alpha \neq \beta$. Again

$$\operatorname{int} \operatorname{pos} C(\alpha) \cap \operatorname{pos} C(\beta)_{\cdot \bar{j}} = \emptyset$$

by Lemma 2.7, so

$$\operatorname{pos} C(\alpha) \cap \operatorname{pos} C(\beta)_{\cdot \bar{j}} \subseteq \partial \operatorname{pos} C(\alpha) .$$

As before (with $\operatorname{pos} C(\mu)$), we must have $\operatorname{pos} C(\alpha) \cap \operatorname{pos} C(\beta)_{\cdot \bar{j}}$ lying in an $(n-1)$-face, say $\operatorname{pos} C(\alpha)_{\cdot \bar{i}}$, of $\operatorname{pos} C(\alpha)$. But

$$\dim[\operatorname{pos} C(\alpha)_{\cdot \bar{i}} \cap \operatorname{pos} C(\beta)_{\cdot \bar{j}}] = n - 1 \tag{2.7}$$

and

$$\dim[\operatorname{pos} C(\alpha)_{\cdot \bar{i}}] = \dim[\operatorname{pos} C(\beta)_{\cdot \bar{j}}] = n - 1 , \tag{2.8}$$

so

$$\operatorname{pos} C(\alpha)_{\cdot \bar{i}} \cup \operatorname{pos} C(\beta)_{\cdot \bar{j}} \subseteq H .$$

Pick $v \neq 0$ orthogonal to H. Then

$$v^T C(\alpha)_{\cdot \bar{i}} = v^T C(\beta)_{\cdot \bar{j}} = 0 .$$

Notice that $C(\alpha)_{\cdot i} \in \operatorname{int} H^-$ (for otherwise, $\operatorname{pos} C(\alpha) \cap \operatorname{int} H^- = \emptyset$). Thus, as $C(\beta)_{\cdot j} \in \operatorname{int} H^+$, and $C(\mu)_{\cdot j} \in H$, we have $i \neq j$. In light of (2.7) and (2.8), there exists a vector $x \in \Re^{n-1}$ such that

$$C(\alpha)_{\cdot i}\, x \in \operatorname{pos} C(\beta)_{\cdot j}\,, \qquad x > 0\,.$$

This completes the proof.

$\square$

Notice, from the proof of sufficiency, that all degenerate cones of a U-matrix must be in $\partial \mathrm{K}(M)$.

2.3 Variations on the Characterisation and Further Results on U-matrices

In the previous section a set of necessary and sufficient conditions was given for a matrix not to be in **U**. These conditions describe **U** as a subclass of $\mathbf{E}_0^{\mathbf{f}}$ by stating exactly what "goes wrong" with an $\mathbf{E}_0^{\mathbf{f}}$-matrix when it is not in **U**. It is of interest to look at other (sufficient) conditions on an $\mathbf{E}_0^{\mathbf{f}}$-matrix that would "force" it out of **U** vis-à-vis (necessary) conditions that would have to hold were the matrix not in **U**. This will give us a better idea of the structure of U-matrices, especially by looking at why other conditions are *not* both necessary and sufficient. With this in mind, we have

THEOREM 2.9 If $M \in \mathbf{E}_0^f \cap \Re^{n\times n}$ and there exist $\alpha, \beta, \gamma \in (\overline{n})$ such that

(i) $\alpha \triangle \beta = \{i\} \neq \{j\} = \alpha \triangle \gamma$,

(ii) $\det M_{\alpha\alpha} = 0$ and $(\det M_{\beta\beta})(\det M_{\gamma\gamma}) > 0$,

(iii) $C(\beta)_{\cdot i}$ and $C(\gamma)_{\cdot j}$ are on opposite sides of $\operatorname{span} C(\alpha)$ [that is, with $x = C(\beta)_{\cdot i}$, $y = C(\gamma)_{\cdot j}$, and $A = C(\alpha)_{\cdot \hat{i}}$, the inequality $x^T(I - A(A^TA)^{-1}A^T)y < 0$ holds],

then $M \notin \mathbf{U}$.

The basic idea here is that if in $\mathrm{K}(M)$ we have two nondegenerate cones "sandwiching in" a degenerate cone, then the matrix cannot be in U.

Proof. By (i) we have that $C(\beta)_{\cdot \hat{i}} = C(\alpha)_{\cdot \hat{i}}$ and $C(\gamma)_{\cdot \hat{j}} = C(\alpha)_{\cdot \hat{j}}$. Since $\det M_{\alpha\alpha} = 0$, we then have a vector $v \neq 0$ such that $v^T C(\beta)_{\cdot \hat{i}} = v^T C(\gamma)_{\cdot \hat{j}} = 0$. By Theorem 2.6 it remains to show that $\operatorname{int}\operatorname{pos} C(\beta)_{\cdot \hat{i}} \cap \operatorname{int}\operatorname{pos} C(\gamma)_{\cdot \hat{j}} \neq \emptyset$. Suppose not. Since $\operatorname{pos} C(\beta)_{\cdot \hat{i}}$ and $\operatorname{pos} C(\gamma)_{\cdot \hat{j}}$ lie in the same $(n-1)$-dimensional subspace, and since $\operatorname{pos} C(\beta)_{\cdot \hat{i}\hat{j}} = \operatorname{pos} C(\gamma)_{\cdot \hat{i}\hat{j}}$, it follows from Lemma 2.8 that $C(\beta)_{\cdot j}$ and $C(\gamma)_{\cdot i}$ lie on opposite sides of $\operatorname{span} C(\beta)_{\cdot \hat{i}\hat{j}}$. (Notice that $C(\alpha)_{\cdot \hat{i}\hat{j}} = C(\beta)_{\cdot \hat{i}\hat{j}} = C(\gamma)_{\cdot \hat{i}\hat{j}}$.) Thus there exists a positive number, θ, a nonzero vector $v \in \operatorname{span} C(\alpha)$, and vectors $z, \bar{z} \in \Re^{n-2}$ for which

$$\begin{aligned} C(\beta)_{\cdot j} &= C(\beta)_{\cdot \hat{i}\hat{j}}\, z + v \\ C(\gamma)_{\cdot i} &= C(\gamma)_{\cdot \hat{i}\hat{j}}\, \bar{z} - \theta v . \end{aligned}$$

From (iii), there exists a positive number, τ, a nonzero vector $w \in \Re^n$, and vectors $x, \bar{x} \in \Re^{n-1}$ for which

$$\begin{aligned} C(\beta)_{\cdot i} &= C(\beta)_{\cdot \hat{i}}\, x + w \\ C(\gamma)_{\cdot j} &= C(\gamma)_{\cdot \hat{j}}\, \bar{x} - \tau w . \end{aligned}$$

Thus,

$$\det C(\beta) = \det[\ C(\beta)_{\cdot\widehat{ij}}\ \mid\ C(\beta)_{\cdot\widehat{ij}}\, z + v\ \mid\ C(\beta)_{\cdot i}\, x + w\ ]$$

where the matrix is represented with column i on the right, column j in the middle, and all other columns on the left. Hence

$$\begin{aligned}
\det C(\beta) &= \det[\ C(\beta)_{\cdot\widehat{ij}} \mid C(\beta)_{\cdot\widehat{ij}}\, z + v \mid w\ ] \\
&= \det[\ C(\beta)_{\cdot\widehat{ij}} \mid v \mid w\ ] \\
&= -\det[\ C(\beta)_{\cdot\widehat{ij}} \mid w \mid v\ ] \\
&= -\frac{1}{\theta\tau} \det[\ C(\gamma)_{\cdot\widehat{ij}} \mid -\tau w \mid -\theta v\ ] \\
&= -\frac{1}{\theta\tau} \det[\ C(\gamma)_{\cdot\widehat{ij}} \mid -\tau w \mid C(\gamma)_{\cdot\widehat{ij}}\, \overline{z} - \theta v\ ] \\
&= -\frac{1}{\theta\tau} \det[\ C(\gamma)_{\cdot\widehat{ij}} \mid C(\gamma)_{\cdot j}\, \overline{x} - \tau w \mid C(\gamma)_{\cdot\widehat{ij}}\, \overline{z} - \theta v\ ] \\
&= -\frac{1}{\theta\tau} \det C(\gamma)\,.
\end{aligned}$$

This contradicts (ii), so our supposition was false and the theorem follows. □

Recall that $\mathbf{P}_0 \subseteq \mathbf{E}_0^f$. If we apply Theorem 2.9 to a $\mathbf{P}_0$-matrix, the inequality "$>$" in (ii) can be replaced by the symbol "$\neq$" and the condition more closely resembles that in Theorem 2.6. Notice, in the proof of Theorem 2.6, that if we knew $M \in \mathbf{P}_0$ and

$$C(\alpha)_{\cdot j} \notin \operatorname{span} C(\beta)_{\cdot\widehat{ij}}\,,$$

we could define

$$\begin{aligned}\overline{\alpha} &= \beta \vartriangle \{j\} \\ \overline{\beta} &= \beta \\ \overline{\gamma} &= \beta \vartriangle \{i,j\}\end{aligned}$$

and then $\overline{\alpha}$, $\overline{\beta}$, and $\overline{\gamma}$ would satisfy the hypotheses of Theorem 2.9. All we've done is verify that $C(\beta)_{\cdot\hat{i}\hat{j}}$ and $C(\alpha)_{\cdot j}$ together constitute a linearly independent set of columns so that, when $C(\alpha)_{\cdot i}$ is adjoined, a nondegenerate complementary cone with the desired position is formed. Thus, we need to have $C(\alpha)_{\cdot j} \in \operatorname{span} C(\beta)_{\cdot\hat{i}\hat{j}}$ and $C(\beta)_{\cdot i} \in \operatorname{span} C(\alpha)_{\cdot\hat{i}\hat{j}}$ to spoil this reasoning. It seems plausible that the conditions of Theorem 2.9 are necessary as well as sufficient for $M \in \mathbf{P}_0 \setminus \mathbf{U}$. However, this is not the case.

EXAMPLE 2.10 Let

$$M = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}.$$

It is easily checked that $M \in \mathbf{P}_0$. The only full complementary cones corresponding to it are $\operatorname{pos} C(\emptyset)$ and $\operatorname{pos} C(\overline{3})$ – i.e., $\operatorname{pos} I$ and $\operatorname{pos} -M$. The hypotheses of Theorem 2.9 cannot be satisfied by this matrix since the index sets β and γ can differ by only two elements. But $M \notin \mathbf{U}$ as, for $q = (1,1,0)^T \in \operatorname{int} \mathrm{K}(M)$, the problem (q, M) has the solutions

$$\begin{aligned}(w^1, z^1) &= (1,1,0,0,0,0) \\ (w^2, z^2) &= (0,0,0,0,1,1).\end{aligned}$$

This example can be used to disprove the necessity of the conditions in Theorem 2.9 because we can construct the two full cones, that "sandwich in" the degenerate cone, so that their index sets differ by more than two

elements. (Clearly they must differ by *at least* two elements.) Thus, we might consider combining conditions (i) and (ii) from Theorem 2.6 with condition (iii) of Theorem 2.9 to get

COROLLARY 2.11 If $M \in \mathbf{E}_0^{\mathbf{f}} \cap \Re^{n\times n}$ and $M \notin \mathbf{U}$ then there exist $\alpha, \beta \in (\overline{n})$ and $i, j \in \overline{n}$ such that

(i) $\alpha \neq \beta$, $i \neq j$,

(ii) $(\det M_{\alpha\alpha})(\det M_{\beta\beta}) \neq 0$ and there exists a nonzero vector $v \in \Re^n$ such that $v^T C(\alpha)_{\cdot i} = v^T C(\beta)_{\cdot j} = 0$,

(iii) $C(\alpha)_{\cdot i}$ and $C(\beta)_{\cdot j}$ are on opposite sides of $\operatorname{span} C(\alpha)_{\cdot \bar{i}} = \operatorname{span} C(\beta)_{\cdot \bar{j}}$ [that is, $(v^T C(\alpha)_{\cdot i})(v^T C(\beta)_{\cdot j}) < 0$].

Proof. This follows immediately from Theorem 2.6, Lemma 2.7 and Lemma 2.8. For (i) and (ii) are from Theorem 2.6, and if $C(\alpha)_{\cdot i}$ and $C(\beta)_{\cdot j}$ were on the same side of $\operatorname{span} C(\alpha)_{\cdot \bar{i}}$, then condition (iii) of Theorem 2.6 and Lemma 2.8 together would imply that $\operatorname{int}\operatorname{pos} C(\alpha) \cap \operatorname{int}\operatorname{pos} C(\beta) \neq \emptyset$, which would contradict Lemma 2.7.

$\square$

To show that these conditions are not sufficient for M not being in U, we have

EXAMPLE 2.12 Let

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

This matrix belongs to $\mathbf{P}_0$, and so is in $\mathbf{E}_0^{\mathbf{f}}$. The only full complementary cones are $\operatorname{pos} I$ and $\operatorname{pos} -M$ which, in this case, meet only at 0. Thus,

$$\operatorname{int} \mathrm{K}(M) = \operatorname{int}\operatorname{pos} I \cup \operatorname{int}\operatorname{pos} -M$$

and clearly $M \in \mathbf{U}$. Yet the three conditions mentioned above are satisfied as

$$\operatorname{span} C(\emptyset)_{.\bar{3}} = \operatorname{span} C(\overline{3})_{.\bar{1}}$$

and $C(\emptyset)_{.3}$ and $C(\overline{3})_{.1}$ lie on opposite sides of $\operatorname{span} C(\emptyset)_{.\bar{3}} = \operatorname{span} C(\overline{3})_{.\bar{1}}$.

Another possible variation of Theorem 2.6 would be to make condition (iii) much stronger. This would clearly preserve the sufficiency of the conditions, giving us

COROLLARY 2.13 Let $M \in \mathbf{E}_0^f \cap \Re^{n\times n}$. If there there exist $\alpha, \beta \in (\overline{n})$ and $i, j \in \overline{n}$ such that

(i) $\alpha \neq \beta$, $i \neq j$,

(ii) $(\det M_{\alpha\alpha})(\det M_{\beta\beta}) \neq 0$ and there exists a nonzero vector $v \in \Re^n$ such that $v^T C(\alpha)_{.\bar{i}} = v^T C(\beta)_{.\bar{j}} = 0$,

(iii) $\operatorname{pos} C(\alpha)_{.\bar{i}} \subseteq \operatorname{pos} C(\beta)_{.\bar{j}}$,

then $M \notin \mathbf{U}$.

□

However, this new condition (iii) is too strong to be necessary, as is shown by

EXAMPLE 2.14 Let

$$M = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Obviously $M \in \mathbf{P}_0 \subseteq \mathbf{E}_0^f$, and the only full complementary cones are $\operatorname{pos} C(\emptyset)$ and $\operatorname{pos} C(\{1,4\})$. They intersect only on the respective faces

pos $C(\emptyset)_{.\bar{4}}$ and pos $C(\{1,4\})_{.\bar{1}}$. (Notice that span $C(\emptyset)_{.\bar{4}} =$ span $C(\{1,4\})_{.\bar{1}}$.) For $x = (1,2,1)^T$ we have

$$C(\{1,4\})_{.\bar{1}}\, x = C(\emptyset)_{.\bar{4}}\, x$$

so the two faces do have (relative) interior points in common. Hence $M \notin \mathrm{U}$ by Theorem 2.6. Now

$$C(\{1,4\})_{.\bar{1}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$C(\emptyset)_{.\bar{4}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

But

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \notin \operatorname{pos}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \notin \operatorname{pos}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

so neither face contains the other.

We now examine the state of affairs for vectors $q \in \partial \operatorname{int} \mathrm{K}(M)$.

THEOREM 2.15 If $M \in \mathbf{E}_0^{\mathfrak{f}} \cap \Re^{n\times n}$ and $q \in \partial \operatorname{int} \mathrm{K}(M)$, then $|\mathrm{sol}(q, M)| = \infty$. (In fact, $\mathrm{sol}(q, M)$ is unbounded.)

Proof. We know q must lie in some $(n-1)$-dimensional face of $\partial \operatorname{int} \mathrm{K}(M)$. Since

$$q \in \partial \operatorname{int} \mathrm{K}(M) \subseteq \bigcup_{\alpha \in (\overline{n})} \{\partial \operatorname{pos} C(\alpha) : \det C(\alpha) \neq 0\},$$

q must belong to an $(n-1)$-dimensional face, $C(\alpha)._{\bar{\imath}}$, of some full complementary cone $\operatorname{pos} C(\alpha)$ such that

$$\dim[\operatorname{pos} C(\alpha)._{\bar{\imath}} \cap \partial \mathrm{K}(M)] = n - 1.$$

The union of all points in $\operatorname{pos} C(\alpha)._{\bar{\imath}} \cap \partial \mathrm{K}(M)$ that are contained in a k-dimensional complementary cone with $k \leq n-2$, that are contained in the boundary of an $(n-1)$-dimensional face of a complementary cone, or that are contained in an $(n-1)$-dimensional face, of a complementary cone, *not* contained in $\operatorname{span} C(\alpha)._{\bar{\imath}}$ is a finite union of sets of dimension $n-2$ or less. Hence, we can find a point $\tilde{q} \in \operatorname{int} \operatorname{pos} C(\alpha)._{\bar{\imath}} \cap \partial \mathrm{K}(M)$ *not* in this union that is arbitrarily close to q. If $\tilde{q} \in \operatorname{pos} C(\beta)._{\bar{\jmath}}$ for some $j \in \overline{n}$, $\beta \neq \alpha$, with $\det C(\beta) \neq 0$, then $\operatorname{pos} C(\beta)._{\bar{\jmath}} \subseteq \operatorname{span} C(\alpha)._{\bar{\imath}}$ and $\tilde{q} \in \operatorname{int} \operatorname{pos} C(\beta)._{\bar{\jmath}}$. So we have either $C(\alpha)._i$ and $C(\beta)._j$ on the same side of $\operatorname{span} C(\alpha)._{\bar{\imath}}$, which by Lemma 2.8 implies that

$$\operatorname{pos} C(\alpha) \cap \operatorname{pos} C(\beta) \neq \emptyset$$

contradicting the assumption that $M \in \mathbf{E}_0^{\mathfrak{f}}$, or else we have $C(\alpha)._i$ and $C(\beta)._j$ on opposite sides of $\operatorname{span} C(\alpha)._{\bar{\imath}}$ which implies that

$$\tilde{q} \in \operatorname{int}[\operatorname{pos} C(\alpha) \cup \operatorname{pos} C(\beta)] \subseteq \operatorname{int} \mathrm{K}(M),$$

contradicting the fact that $\tilde{q} \in \partial \mathrm{K}(M)$. So $\tilde{q}$ is contained in only one $(n-1)$-dimensional face of one full cone. From Lemma 3.2 of Saigal (1972a), we see that $\tilde{q}$ must be contained in some complementary cone, $\operatorname{pos} C(\beta)$, where $C(\beta)x = 0$ for some nonzero $x \geq 0$. As there are finitely many such cones, and as $\tilde{q}$ was arbitrarily close to q, we can find a sequence $q_\nu \to q$ in some such cone. As all such cones are closed, we may assume without loss of generality that $q \in \operatorname{pos} C(\beta)$. Thus $q = C(\beta)y$ for some $y \geq 0$, and for each $\lambda \geq 0$, $y + \lambda x$ will give us a different solution to (q, M).

□

Theorem 2.15 explains why we must define $\mathbf{U}$ with respect to the interior of $\mathrm{K}(M)$, rather than all of $\mathrm{K}(M)$. If (q, M) has a unique solution for $q \in \mathrm{K}(M)$, then certainly $M \in \mathbf{U}$. But Theorem 2.15 then requires that $\partial \operatorname{int} \mathrm{K}(M) = \emptyset$. Thus we must have $\operatorname{int} \mathrm{K}(M) = \Re^n$ thus $M \in \mathbf{Q}$. However, $\mathbf{U} \cap \mathbf{Q} = \mathbf{P}$ which gives us nothing new. In fact, the proof of Theorem 2.15 shows that if $q \in \partial \operatorname{int} \mathrm{K}(M)$ then q is in a strongly degenerate cone. Thus we have

Corollary 2.16 If $M \in \mathbf{E}_0^{\mathbf{f}}$, then $\partial \operatorname{int} \mathrm{K}(M)$ is contained in the union of the strongly degenerate cones.

□

Corollary 2.17 If $M \in \mathbf{U}$ and M is nondegenerate or weakly degenerate, then $M \in \mathbf{P}$.

□

2.4 $\mathbf{E}_0^f \cap \mathbf{Q}_0$ -matrices and $\mathbf{U} \cap \mathbf{Q}_0$ -matrices

In this section we confine our attention to those matrices within $\mathbf{E}_0^f$ and $\mathbf{U}$ which are also in $\mathbf{Q}_0$. (Recall that $M \in \mathbf{Q}_0$ if and only if $\mathrm{K}(M)$ is convex.) We start off with a lemma used to prove the next (familiar) theorem. It expresses the underlying structure of $\partial\,\mathrm{K}(M)$ for $\mathbf{U} \cap \mathbf{Q}_0$ -matrices.

LEMMA 2.18 Suppose $M \in \mathbf{U} \cap \mathbf{Q}_0 \cap \Re^{n\times n}$ and let $\operatorname{pos} C(\alpha)$ be a full complementary cone relative to M. Define the index set $\beta = \alpha \triangle \{i\}$. Then $\operatorname{span} C(\alpha)_{\cdot \bar{i}}$ is a supporting (boundary) hyperplane of $\mathrm{K}(M)$ if and only if $C(\beta)_{\cdot i}$ lies in $\operatorname{span} C(\alpha)_{\cdot \bar{i}}$.

Proof. If $C(\beta)_{\cdot i} \in C(\alpha)_{\cdot \bar{i}}$, then $\operatorname{pos} C(\beta)$ is a degenerate cone. Therefore $\operatorname{pos} C(\beta) \in \partial\,\mathrm{K}(M)$ as $M \in \mathbf{U}$. Since $\operatorname{pos} C(\beta) \subseteq \operatorname{span} C(\alpha)_{\cdot \bar{i}}$, we see that

$$\dim[\operatorname{span} C(\alpha)_{\cdot \bar{i}} \cap \partial\,\mathrm{K}(M)] = n-1\,.$$

Thus $\operatorname{span} C(\alpha)_{\cdot \bar{i}}$ is a supporting hyperplane of the finite convex cone $\mathrm{K}(M)$.

Conversely, suppose $C(\beta)_{\cdot i} \notin \operatorname{span} C(\alpha)_{\cdot \bar{i}}$. If $C(\alpha)_{\cdot i}$ and $C(\beta)_{\cdot i}$ were on the same side of $\operatorname{span} C(\alpha)_{\cdot \bar{i}}$, then by Lemma 2.8, the interiors of the full complementary cones $\operatorname{pos} C(\alpha)$ and $\operatorname{pos} C(\beta)$ would intersect. This contradicts that $M \in \mathbf{E}_0^f$. Thus, $C(\alpha)_{\cdot i}$ and $C(\beta)_{\cdot i}$ are on opposite sides of $\operatorname{span} C(\alpha)_{\cdot \bar{i}}$. Hence we have

$$\operatorname{int}\operatorname{pos} C(\alpha)_{\cdot \bar{i}} \subseteq \operatorname{int}[\operatorname{pos} C(\alpha) \cup \operatorname{pos} C(\beta)] \subseteq \operatorname{int}\mathrm{K}(M)\,,$$

so $\operatorname{span} C(\alpha)_{\cdot \bar{i}}$ cannot be a supporting hyperplane of $\mathrm{K}(M)$.

□

Without the assumption that $M \in \mathbf{U}$ we find that *both* directions in Lemma 2.18 fail to hold. The matrix

$$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

see Figure 2.3 again, is in $\mathbf{Q}_0 \cap \mathbf{E}_0^{\mathbf{f}}$ but is not in $\mathbf{U}$. As always, $\operatorname{pos} C(\emptyset)$ is a nondegenerate complementary cone, and $C(\{2\})_{\cdot 2} \in \operatorname{span} C(\emptyset)_{\cdot \hat{2}}$. But $\operatorname{span} C(\emptyset)_{\cdot \hat{2}}$ is *not* a supporting hyperplane to $\mathrm{K}(M)$. For the other direction, consider

$$M = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

This matrix is also in $\mathbf{Q}_0$ but not in $\mathbf{U}$. In this instance, $\mathrm{K}(M) = \Re_+^2$. Lemma 2.18 would make each boundary hyperplane contain a degenerate complementary cone. But this is clearly not the case.

Notice that the second of these matrices is not in $\mathbf{E}_0^{\mathbf{f}}$, and cannot be as the second part of the proof only needed $M \in \mathbf{E}_0^{\mathbf{f}}$. Notice, also, that the first of these two matrices belongs to $\mathbf{P}_0$, but not the second. In fact we prove

COROLLARY 2.19 Suppose $M \in \mathbf{P}_0 \cap \mathbf{Q}_0 \cap \Re^{n \times n}$ and let $\operatorname{pos} C(\alpha)$ be a full complementary cone relative to M. Define the index set $\beta = \alpha \,\triangle\, \{i\}$. If $\operatorname{span} C(\alpha)_{\cdot \hat{\imath}}$ is a supporting (boundary) hyperplane of $\mathrm{K}(M)$, then $C(\beta)_{\cdot i} \in \operatorname{span} C(\alpha)_{\cdot \hat{\imath}}$.

Proof. Suppose $C(\beta)_{\cdot i} \notin \operatorname{span} C(\alpha)_{\cdot \hat{\imath}}$. If $C(\alpha)_{\cdot i}$ and $C(\beta)_{\cdot i}$ are on the same side of $\operatorname{span} C(\alpha)_{\cdot \hat{\imath}} = \operatorname{span} C(\beta)_{\cdot \hat{\imath}}$, then $\det C(\alpha)$ and $\det C(\beta)$ are not zero and have the same sign. Thus $(\det M_{\alpha\alpha})(\det M_{\beta\beta}) < 0$. This is impossible when $M \in \mathbf{P}_0$. Thus, $C(\alpha)_{\cdot i}$ and $C(\beta)_{\cdot i}$ are on opposite sides of $\operatorname{span} C(\alpha)_{\cdot \hat{\imath}}$. As in the proof of Lemma 2.18, we have

$\operatorname{int}\operatorname{pos} C(\alpha)_{\cdot i} \subseteq \operatorname{int}[\operatorname{pos} C(\alpha) \cup \operatorname{pos} C(\beta)] \subseteq \operatorname{int} \mathrm{K}(M)$, so $\operatorname{span} C(\alpha)_{\cdot i}$ cannot be a supporting hyperplane of $\mathrm{K}(M)$.

□

With Lemma 2.18, we can show that for $M \in \mathbf{Q}_0$ the conditions of Corollary 2.11 are sufficient as well as necessary for $M \in \mathbf{E}_0^f$ not to be in $\mathbf{U}$. We have

THEOREM 2.20 If $M \in \mathbf{E}_0^f \cap \mathbf{Q}_0 \cap \Re^{n \times n}$, then $M \notin \mathbf{U}$ if and only if there exist $\alpha, \beta \in (\overline{n})$ and $i, j \in \overline{n}$ such that

(i) $\alpha \neq \beta$, $i \neq j$,

(ii) $(\det M_{\alpha\alpha})(\det M_{\beta\beta}) \neq 0$ and there exists a nonzero vector $v \in \Re^n$ such that $v^T C(\alpha)_{\cdot i} = v^T C(\beta)_{\cdot j} = 0$,

(iii) $C(\alpha)_{\cdot i}$ and $C(\beta)_{\cdot j}$ are on opposite sides of $\operatorname{span} C(\alpha)_{\cdot i} = \operatorname{span} C(\beta)_{\cdot j}$ [that is, $(v^T C(\alpha)_{\cdot i})(v^T C(\beta)_{\cdot j}) < 0$].

Proof. The necessity part of this theorem follows from Corollary 2.11. Now suppose that the conditions are satisfied. We know that $\operatorname{pos} C(\alpha)$ is a full complementary cone. As $i \neq j$ and $\operatorname{span} C(\alpha)_{\cdot i} = \operatorname{span} C(\beta)_{\cdot j}$ we would have $C(\alpha)_{\cdot i} \neq C(\beta)_{\cdot i}$ and $C(\beta)_{\cdot i} \in \operatorname{span} C(\alpha)_{\cdot i}$. So if M were a U-matrix, Lemma 2.18 would imply that $\operatorname{span} C(\alpha)_{\cdot i}$ is a supporting hyperplane of $\mathrm{K}(M)$. But this is impossible if $C(\alpha)_{\cdot i}$ and $C(\beta)_{\cdot j}$ lie on opposite sides of $\operatorname{span} C(\alpha)_{\cdot i}$.

□

Condition (iii) in Theorem 2.20 is non-trivial. Figure 2.4 shows $\mathrm{K}(M)$ for the matrix

$$M = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

which is in $\mathbf{E}_0^{\mathbf{f}} \cap \mathbf{Q}_0$. This matrix satisfies (i) and (ii) with $(\alpha, \beta, i, j) = (\emptyset, \overline{2}, 1, 2)$. However, (iii) is not satisfied, and, indeed, $M \in \mathbf{U}$.

Notice that Theorem 2.20 implies Example 2.12 must have used a matrix M not in $\mathbf{Q}_0$ which, in fact, it did. However, Example 2.10 used a $\mathbf{Q}_0$-matrix so we cannot strengthen Theorem 2.9 for $\mathbf{E}_0^{\mathbf{f}} \cap \mathbf{Q}_0$ -matrices. Example 2.14 does not use a $\mathbf{Q}_0$-matrix, but we still cannot strengthen Corollary 2.13 as seen by

EXAMPLE 2.21 Let

$$M = \begin{bmatrix} 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The full complementary cones are $C(\emptyset)$, $C(\{3\})$, $C(\{1,4\})$ and $C(\{1,2,4\})$. Suppose, for the sake of contradiction, that $M \notin \mathbf{E}_0$. Then there is a nonzero $x \geq 0$, so that for any $i \in \overline{4}$, if $x_i > 0$, then $(Mx)_i < 0$. As we will always have $(Mx)_3, (Mx)_4 \geq 0$, we must have $x_3 = x_4 = 0$. But then we will have $(Mx)_2 \geq 0$, thus requiring that $x_2 = 0$. Cumulatively these conditions will cause $(Mx)_1$ to be nonnegative, leading us to conclude that $x_1 = 0$, giving a contradiction. Thus $M \in \mathbf{E}_0$. M has three non-trivial principal transforms which correspond to block pivots on $M_{\alpha\alpha}$ where α can be $\{3\}$, $\{1,4\}$, or $\{1,2,4\}$, and are, respectively,

$$\begin{bmatrix} 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix}.$$

Similar arguments will show that these three matrices are all in $\mathbf{E}_0$. Thus

$M \in \mathbf{E}_0^f$. Now, it is clear that $\mathrm{K}(M) \subseteq \mathrm{pos}[\, I \mid -M\,]$. Given the following fact (see, for example, Proposition 4.2 of Doverspike and Lemke (1979))

$$M \in \mathbf{Q}_0 \quad \Leftrightarrow \quad \mathrm{pos}[\, I \mid -M\,] = \bigcup \{\, \mathrm{pos}\, C(\alpha) : \det C(\alpha) \neq 0 \,\},$$

and noting that

$$\mathrm{pos}[\, I \mid -M\,] = \{\, x \in \Re^4 : x_1, x_2, x_3 \geq 0 \text{ and } x_1, x_2 \geq -x_3 \,\},$$

$$\begin{aligned}
\mathrm{pos}\, C(\emptyset) &= \{\, x \in \Re^4 : x_1, x_2, x_3, x_4 \geq 0 \,\}, \\
\mathrm{pos}\, C(\{\,3\,\}) &= \{\, x \in \Re^4 : x_1, x_2, -x_3, x_4 \geq 0 \text{ and } x_1, x_2 \geq -x_3 \,\}, \\
\mathrm{pos}\, C(\{\,1,4\,\}) &= \{\, x \in \Re^4 : x_1, x_2, -x_4 \geq 0 \text{ and } x_2 \geq x_1 \geq -x_3 \,\}, \\
\mathrm{pos}\, C(\{\,1,2,4\,\}) &= \{\, x \in \Re^4 : x_1, x_2, -x_4 \geq 0 \text{ and } x_1 \geq x_2 \geq -x_3 \,\},
\end{aligned}$$

we find that $M \in \mathbf{Q}_0$. As in Example 2.14, for $x = (1,2,1)^T$, we have that $\mathrm{span}\, C(\{\,1,4\,\})_{\cdot\hat{1}} = \mathrm{span}\, C(\emptyset)_{\cdot\hat{4}}$ and $C(\{\,1,4\,\})_{\cdot\hat{1}}\, x = C(\emptyset)_{\cdot\hat{4}}\, x$; thus $M \notin \mathbf{U}$. However, there are only four candidiates for the 4-tuple (α, β, i, j) in the conditions of Theorem 2.6, and checking them shows that, for each, we have some $q \in \mathrm{pos}\, C(\alpha)_{\cdot\hat{\imath}} \setminus \mathrm{pos}\, C(\beta)_{\cdot\hat{\jmath}}$ and some $\tilde{q} \in \mathrm{pos}\, C(\beta)_{\cdot\hat{\jmath}} \setminus \mathrm{pos}\, C(\alpha)_{\cdot\hat{\imath}}$,

$(C(\emptyset), C(\{\,1,4\,\}), 4, 1)$	$q = (2,1,0,0)^T$	$\tilde{q} = (1,1,-1,0)^T$
$(C(\emptyset), C(\{\,1,2,4\,\}), 4, 1)$	$q = (1,2,0,0)^T$	$\tilde{q} = (1,1,-1,0)^T$
$(C(\{\,3\,\}), C(\{\,1,4\,\}), 4, 1)$	$q = (2,1,0,0)^T$	$\tilde{q} = (1,1,1,0)^T$
$(C(\{\,3\,\}), C(\{\,1,2,4\,\}), 4, 1)$	$q = (1,2,0,0)^T$	$\tilde{q} = (1,1,1,0)^T$.

Hence, M is an example showing that Corollary 2.13 cannot be strengthen to say that its three conditions are *necessary* for a matrix in $\mathbf{E}_0^f \cap \mathbf{Q}_0$ not to be in $\mathbf{U}$.

We now come to a result which says that when $\mathrm{K}(M)$ is convex and (q, M) has a unique solution for all $q \in \mathrm{int}\, \mathrm{K}(M)$, the matrix M cannot

have any negative principal minors. The proof sheds light on the conical structure of $\mathbf{Q}_0 \cap \mathbf{U}$-matrices.

THEOREM 2.22 $\mathbf{Q}_0 \cap \mathbf{U} \subseteq \mathbf{P}_0$.

Proof. Let $M \in \mathbf{Q}_0 \cap \mathbf{U} \cap \Re^{n \times n}$. There are two cases. If $M \in \mathbf{Q}$, then $M \in \mathbf{Q} \cap \mathbf{U} = \mathbf{P} \subseteq \mathbf{P}_0$. Assume therefore that $M \in \mathbf{Q}_0 \setminus \mathbf{Q}$. Thus, $\mathrm{K}(M) \neq \Re^n$. Suppose we have a collection of index sets $\alpha_1, \ldots, \alpha_k \in (\overline{n})$ for which

$$\det M_{\alpha_j \alpha_j} > 0, \qquad j = 1, \ldots, k. \tag{2.9}$$

We know $k \geq 1$ since $\alpha_1 = \emptyset$ belongs to the collection. Now consider

$$C_k = \bigcup_{j=1}^{k} \operatorname{pos} C(\alpha_j),$$

and suppose $C_k \neq \mathrm{K}(M)$.

As $M \in \mathbf{Q}_0$, $\mathrm{K}(M)$ is a closed convex finite cone. The cone C_k is closed and polyhedral; by our assumption, it is a proper subset of $\mathrm{K}(M)$. Thus there must exist a point $q \in \operatorname{int} \mathrm{K}(M) \setminus C_k$. Let $p \in \operatorname{int} C_k$. (Note: C_k contains $\Re^n_+$ and so has a nonempty interior.) Let

$$L = \{ r : r = (1-\lambda)p + \lambda q, \quad 0 \leq \lambda \leq 1 \},$$

and so $\operatorname{int} L = \{ r : r = (1-\lambda)p + \lambda q, \quad 0 < \lambda < 1 \}$. As $L \cap \operatorname{int} \mathrm{K}(M) \neq \emptyset$, $\mathrm{K}(M)$ is convex, and (hence) $L \subseteq \mathrm{K}(M)$, we have

$$\operatorname{int} L \subseteq \operatorname{int} \mathrm{K}(M) \qquad \text{and} \qquad \operatorname{int} L \cap \partial C_k \neq \emptyset.$$

Thus,

$$\partial C_k \cap \operatorname{int} \mathrm{K}(M) \neq \emptyset.$$

Now ∂C_k is $(n-1)$-dimensional and contained in

$$\bigcup_{\substack{j\in\bar{k}\\ i\in\bar{n}}} \operatorname{pos} C(\alpha_j)_{\cdot\hat{i}}\,.$$

Hence as the union of the boundaries of the $\operatorname{pos} C(\alpha_j)_{\cdot\hat{i}}$ is $(n-2)$-dimensional, and as we can slightly perturb the position of the point p we selected and still keep it within $\operatorname{int} C_k$, then we may assume that

$$\operatorname{int}\operatorname{pos} B \cap \partial C_k \cap \operatorname{int} \mathrm{K}(M) \neq \emptyset \tag{2.10}$$

where $B = C(\alpha_j)_{\cdot\hat{n}} \in \Re^{n\times(n-1)}$ for some α_j in the given set satisfying (2.9). Let

$$\beta_j = \alpha_j \vartriangle \{n\}.$$

Now $\operatorname{pos} B \subseteq \operatorname{pos} C(\beta_j)$. If $\det C(\beta_j) = 0$, then there exists a point $\tilde{q} \in \operatorname{int}\operatorname{pos} C(\beta_j) \cap \operatorname{int} \mathrm{K}(M)$, and $(\tilde{q}, M)$ has infinitely many solutions. This contradicts the hypothesis that $M \in \mathbf{U}$. Thus $\det C(\beta_j) \neq 0$, and accordingly, $\det M_{\beta_j\beta_j} \neq 0$. If $\det M_{\beta_j\beta_j} < 0$, then as $\det M_{\alpha_j\alpha_j} > 0$, we have $(\det C(\alpha_j))(\det C(\beta_j)) > 0$ implying that $C(\alpha_j)_{\cdot n}$ and $C(\beta_j)_{\cdot n}$ lie on the same side of $\operatorname{span} B$. Thus, by Lemma 2.8, $\operatorname{int}\operatorname{pos} C(\alpha_j) \cap \operatorname{int}\operatorname{pos} C(\beta_j) \neq \emptyset$, which contradicts the assumption that $M \in \mathbf{U}$. Thus $\det M_{\beta_j\beta_j} > 0$, and we have $I_{\cdot n}$ and $-M_{\cdot n}$ lying on opposite sides of $\operatorname{span} B$. Hence

$$\operatorname{int}\operatorname{pos} B \subseteq \operatorname{int}[\operatorname{pos} C(\alpha_j) \cup \operatorname{pos} C(\beta_j)]\,.$$

From this and (2.10) we have $\beta_j \notin \{\alpha_1, \ldots, \alpha_k\}$. Let $\alpha_{k+1} = \beta_j$ and adjoin it to the collection of known index sets for which the corresponding principal minor is positive. We repeat this construction until l index sets are

found and

$$C_l = \bigcup_{j=1}^{l} \operatorname{pos} C(\alpha_j) = \mathrm{K}(M) .$$

If $\beta \in (\bar{n})$ and $\beta \notin \{\alpha_1, \ldots, \alpha_l\}$, then $\det M_{\beta\beta} = 0$; otherwise

$$\operatorname{int} \operatorname{pos} C(\beta) \cap \operatorname{int} \mathrm{K}(M) \neq \emptyset$$

and this implies there exists α_j $(1 \leq j \leq l)$ such that

$$\operatorname{int} \operatorname{pos} C(\beta) \cap \operatorname{int} \operatorname{pos} C(\alpha_j) \neq \emptyset$$

which contradicts our assumption that $M \in \mathbf{U}$.

□

The next theorem sharpens the ideas concerning the structure of $\partial \mathrm{K}(M)$, for $M \in \mathbf{Q}_0 \cap \mathbf{U}$, that we developed in the proof of the last theorem.

THEOREM 2.23 If $M \in (\mathbf{Q}_0 \setminus \mathbf{Q}) \cap \mathbf{U} \cap \Re^{n \times n}$, then there exists a nonnegative $m \times n$ matrix A such that

$$\mathrm{K}(M) = \{ q : Aq \geq 0 \},$$

and the number m is minimal. Moreover, if

$$\alpha_k = \operatorname{supp} A_{k\cdot} \qquad \text{for all } k \in \bar{m}$$

then $\det M_{\alpha_k \alpha_k} = 0$. If $\det M_{\beta\beta} = 0$, for some $\beta \in (\bar{n})$, then there exists $k \in \bar{m}$ such that $\alpha_k \subseteq \beta$.

Proof. From Theorem 2.22, we know that $M \in \mathbf{P}_0 \setminus \mathbf{P}$. The cone $\mathrm{K}(M)$ being convex and finitely generated can be expressed as a polyhedral

convex cone (see Weyl (1935)). Thus there exists a matrix $A \in \Re^{m\times n}$ such that

$$\mathrm{K}(M) = \{\, q : Aq \geq 0 \,\}.$$

The matrix A can be chosen so that none of its rows is redundant. Since $\Re^n_+ \subseteq \mathrm{K}(M)$, it follows that $A \geq 0$ (and $A_{k\cdot} \neq 0$ for all $k \in \overline{m}$). Each of the hyperplanes

$$H(A_{k\cdot}) = \{\, x \in \Re^n : A^T_{k\cdot}\, x = 0 \,\} \qquad k \in \overline{m}$$

is the boundary of a half-space

$$H^+(A_{k\cdot}) = \{\, x \in \Re^n : A^T_{k\cdot}\, x \geq 0 \,\} \qquad k \in \overline{m}$$

and has an $(n-1)$-dimensional intersection with $\partial\,\mathrm{K}(M)$. For each $k \in \overline{m}$, there exists an $\alpha \in (\overline{n})$ such that

$$\dim[\operatorname{pos} C(\alpha) \cap H(A_{k\cdot})] = n - 1\,.$$

If $\det M_{\alpha\alpha} = 0$, then

$$\operatorname{pos} C(\alpha) \subseteq H(A_{k\cdot}). \tag{2.11}$$

If $\det M_{\alpha\alpha} \neq 0$, then by (2.11) there must exist an index $i \in \overline{n}$ such that

$$\dim[\operatorname{pos} C(\alpha)_{\cdot\bar{i}} \cap H(A_{k\cdot})] = n - 1\,, \tag{2.12}$$

and

$$C(\alpha)_{\cdot i} \notin H(A_{k\cdot}).$$

Let $\beta = \alpha \vartriangle \{\, i \,\}$. If $\det M_{\beta\beta} \neq 0$, as $M \in \mathbf{P}_0$, we have

$$(\det C(\alpha))(\det C(\beta)) < 0\,.$$

So $C(\alpha)_{\cdot i}$ and $C(\beta)_{\cdot i}$ lie on opposite sides of $\operatorname{span} C(\alpha)_{\cdot \hat{\imath}} = \operatorname{span} C(\beta)_{\cdot \hat{\imath}}$. Thus

$$\operatorname{int} \operatorname{pos} C(\alpha)_{\cdot \hat{\imath}} \subseteq \operatorname{int}[\operatorname{pos} C(\alpha) \cup \operatorname{pos} C(\beta)] \subseteq \operatorname{int} \mathrm{K}(M),$$

which contradicts (2.12). So, $\det M_{\beta\beta} = 0$.

Hence for every $k \in \overline{m}$, there exists a $\beta_k \in (\overline{m})$, with $\det M_{\beta_k\beta_k} = 0$, and such that (2.11) holds with $\alpha = \beta_k$. Then

$$j \in \alpha_k \quad \Rightarrow \quad I_{\cdot k} \notin H(A_{k\cdot}) \quad \Rightarrow \quad I_{\cdot k} \notin \operatorname{pos} C(\beta_k) \quad \Rightarrow \quad -M_{\cdot k} \in H(A_{k\cdot}),$$

and

$$j \notin \alpha_k \quad \Rightarrow \quad I_{\cdot k} \in H(A_{k\cdot}).$$

Thus, the columns of $C(\alpha_k)$ are all in $H(A_{k\cdot})$ which implies $\det M_{\alpha_k\alpha_k} = 0$. In fact, if $\beta \in (\overline{n})$ and $\det M_{\beta\beta} = 0$, then $\operatorname{pos} C(\beta) \subseteq \partial \mathrm{K}(M)$, so $\operatorname{pos} C(\beta) \subseteq H(A_{k\cdot})$ for some k, and as above, $j \in \alpha_k$ implies $I_{\cdot k} \notin H(A_{k\cdot})$, so $C(\beta)_{\cdot j} = -M_{\cdot j}$. This implies $\alpha_k \subseteq \beta$.

□

We now examine a situation which could be viewed as a partial converse to Theorem 2.22. It involves matrices belonging to a special subclass of $\mathbf{P}_0$. We shall show that these matrices belong to $\mathbf{U}$ and that they give rise to cones $\mathrm{K}(M)$ of a special form. To this end, we introduce

DEFINITION 2.24 If $M \in \mathbf{P}_0 \cap \Re^{n \times n}$, then $M \in \mathbf{P}_1$ if and only if there exists a unique index set $\alpha \in (\overline{n})$ such that $\det M_{\alpha\alpha} = 0$.

Thus, $M \in \mathbf{P}_1$ if and only if it has nonnegative principal minors precisely one of which is zero. A $\mathbf{P}_1$-matrix may or may not belong to $\mathbf{Q}$. For instance

$$M = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \in \mathbf{P}_1 \cap \mathbf{Q},$$

see Figure 2.5, whereas

$$M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \in \mathbf{P}_1 \setminus \mathbf{Q},$$

see Example 2.2. In the former case, the matrix does not belong to U, but in the later case it does.

THEOREM 2.25 If $M \in (\mathbf{P}_1 \setminus \mathbf{Q}) \cap \Re^{n\times n}$, then $M \in \mathbf{U}$, and $\mathrm{K}(M)$ is a half-space. Furthermore, let $a \in \Re^n$ be the normal to the hyperplane $\partial\,\mathrm{K}(M)$. If $\det M_{\alpha\alpha} = 0$, then a can be chosen so that $a_\alpha > 0$ and $a_{\bar\alpha} = 0$.

Proof. Let $\mathrm{K}_f(M)$ be the union of the *full* complementary cones associated with M. Then $\partial\mathrm{K}_f(M)$ is contained in the union of the boundaries of the full cones. Suppose $\operatorname{pos} C(\alpha)$ is a full complementary cone and

$$\dim[\partial\mathrm{K}_f(M) \cap \operatorname{pos} C(\alpha)_{\cdot\hat{\imath}}] = n - 1.$$

Let $\beta = \alpha \,\triangle\, \{i\}$. If $\operatorname{pos} C(\beta)$ is a full cone, we may ask: where is $C(\beta)_{\cdot i}$ with respect to $\operatorname{span} C(\alpha)_{\cdot\hat{\imath}} = \operatorname{span} C(\beta)_{\cdot\hat{\imath}}$? If $C(\beta)_{\cdot i}$ is on the same side of $\operatorname{span} C(\alpha)_{\cdot\hat{\imath}}$ as $C(\alpha)_{\cdot i}$, then $(\det C(\alpha))(\det C(\beta)) > 0$ giving us the contradiction that $(\det M_{\alpha\alpha})(\det M_{\beta\beta}) < 0$. If $C(\beta)_{\cdot i}$ is on the opposite side, then

$$\operatorname{int} \operatorname{pos} C(\alpha)_{\cdot\hat{\imath}} \subseteq \operatorname{int}[\operatorname{pos} C(\alpha) \cup \operatorname{pos} C(\beta)],$$

so

$$\dim[\partial\mathrm{K}_f(M) \cap \operatorname{pos} C(\alpha)_{\cdot\hat{\imath}}] \leq n - 2,$$

a contradiction. Hence $\operatorname{pos} C(\beta)$ is degenerate and contains $C(\alpha)_{\cdot\hat{\imath}}$. Therefore $\partial\mathrm{K}_f(M)$ is contained in the union of the degenerate complementary

cones. But, by hypothesis, there is only one degenerate complementary cone. Since $M \notin \mathbf{Q}$, we have $\partial \mathrm{K}_{\mathrm{f}}(M) \neq \emptyset$. Thus, $\partial \mathrm{K}_{\mathrm{f}}(M)$ is contained in this one degenerate complementary cone.

Let $L = \{ x : a^T x = 0 \}$ be the affine hull of this degenerate complementary cone. (Both are $(n-1)$-dimensional.) Being the boundary of an n-dimensional polyhedral cone contained in L, $\partial \mathrm{K}_{\mathrm{f}}(M)$ cannot have a boundary relative to L. Hence $\partial \mathrm{K}_{\mathrm{f}}(M) = L$, and $\mathrm{K}(M)$ is a half-space $\{ x : a^T x \geq 0 \}$ with $0 \neq a \geq 0$ as $\Re^n_+ \subseteq \mathrm{K}(M)$.

If $\det M_{\alpha\alpha} = 0$, then $\operatorname{pos} C(\alpha)$ is the only degenerate complementary cone. Thus $I_{\cdot i} \notin L$ if and only if $i \in \alpha$. This implies $a_\alpha > 0$ and $a_{\bar{\alpha}} = 0$.

Moreover $M \in \mathbf{P}_0 \subseteq \mathbf{E}^{\mathrm{f}}_0$, and the fact that the only degenerate cone is $\partial \mathrm{K}_{\mathrm{f}}(M)$ forces the three conditions in Theorem 2.6 to fail to be satisfied, so we have $M \in \mathbf{U}$.

□

As final remark before leaving this chapter, lest the impression be given that $\mathbf{E}^{\mathrm{f}}_0$ is made up of only matrices that are $\mathbf{P}_0$, $\mathbf{U}$, or $\mathbf{Q}_0$, we give an example of a matrix that is in $\mathbf{E}^{\mathrm{f}}_0 \setminus (\mathbf{P}_0 \cup \mathbf{U} \cup \mathbf{Q}_0)$.

EXAMPLE 2.26 Let

$$M = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Clearly $M \notin \mathbf{P}_0$ as $\det M_{\alpha\alpha} < 0$ for $\alpha = \{2, 4\}$. M has exactly four nondegenerate cones: $C(\emptyset)$, $C(\bar{n})$, $C(\{1,3\})$, and $C(\{2,4\})$. Each is a different orthant in $\Re^4$, so the interiors of these four cones are pair-wise disjoint, and

hence $M \in \mathbf{E}_0^{\mathfrak{f}}$. However, with $(\alpha, \beta, i, j) = (\emptyset, \{1, 3\}, 3, 1)$ we can satisfy the conditions of Theorem 2.6 – in fact, $\operatorname{pos} C(\emptyset)_{\cdot\hat{3}} = \operatorname{pos} C(\{1,3\})_{\cdot\hat{1}}$ – and so $M \notin \mathbf{U}$. Finally, we have $(0,2,0,0)^T \in \mathrm{K}(M)$ and $(0,0,0,-2)^T \in \mathrm{K}(M)$, but $(0,1,0,-1)^T \notin \mathrm{K}(M)$, so $\mathrm{K}(M)$ is not convex. Hence $M \notin \mathbf{Q}_0$. Thus $M \in \mathbf{E}_0^{\mathfrak{f}} \setminus (\mathbf{P}_0 \cup \mathbf{U} \cup \mathbf{Q}_0)$ as claimed.

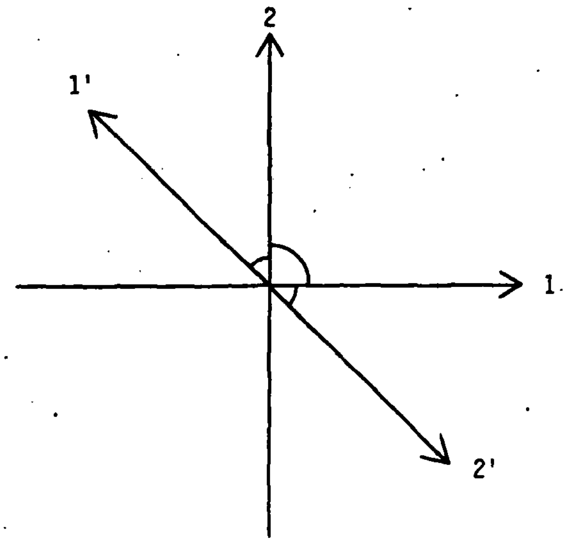

Figure 2.1

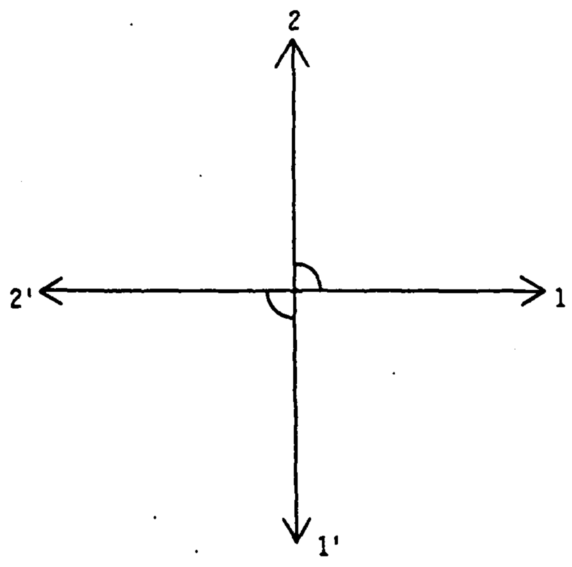

Figure 2.2

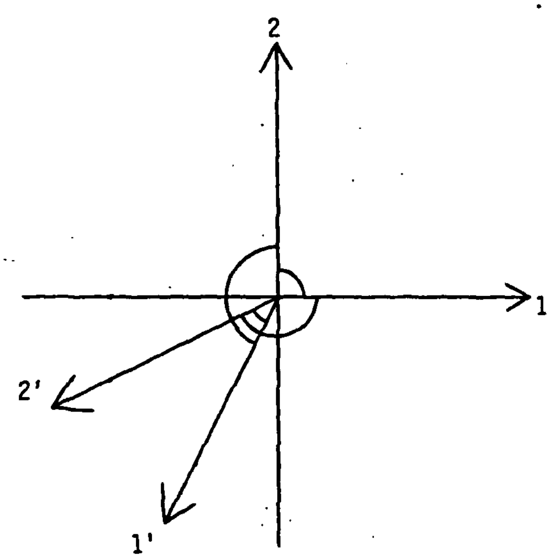

Figure 2.3

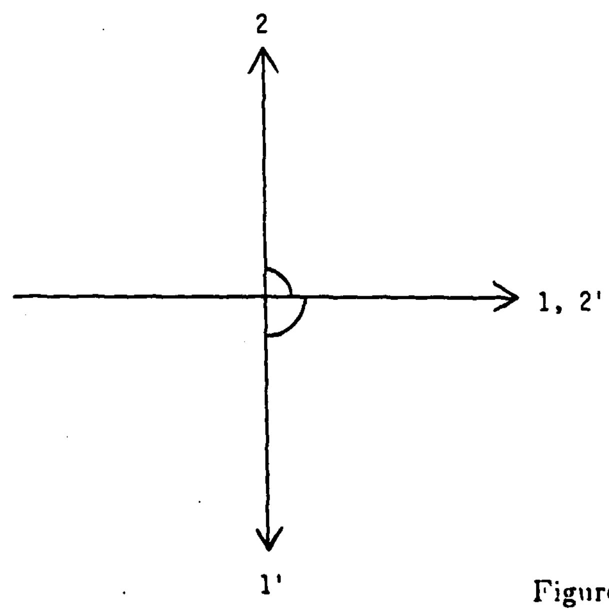

Figure 2.4

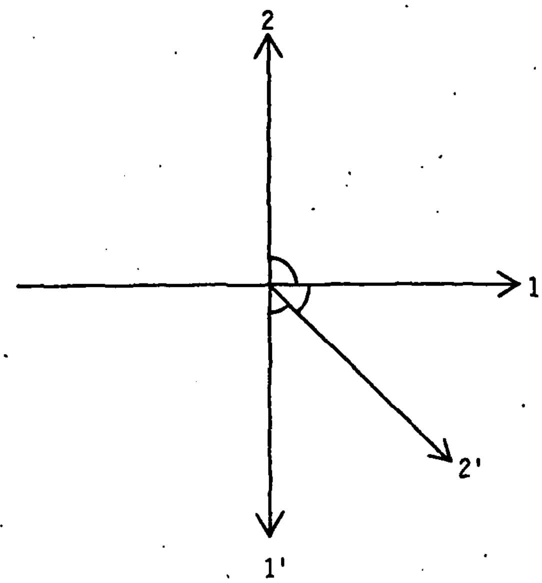

Figure 2.5

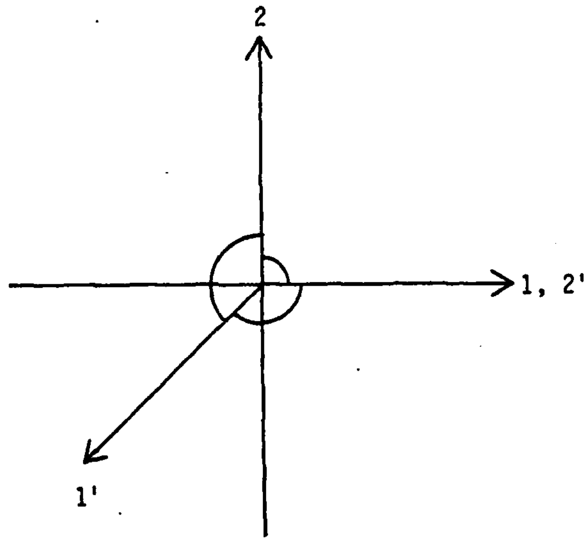

Figure 2.6

CHAPTER 3.
INS-MATRICES: CHARACTERIZATION RESULTS

3.1 Introduction to INS-matrices

We have now defined and studied the class **U** which generalizes the class **P**. We are led to wonder about possible larger classes containing **U**. As before, we must decide what properties we wish this larger class to inherit from **U** and what properties we wish to relax. The one essential property of **U** is the uniqueness of the solution to (q, M) where q is in the interior of $\mathrm{K}(M)$. However, the main properties of the combinatorial and geometric structure of $\mathrm{K}(M)$, that is peculiar to those $M \in \mathbf{U}$, is derived more from having the same number of solutions everywhere within the interior of $\mathrm{K}(M)$ than from that number being, in particular, one. With this in mind, we focus attention on understanding this structure. We have

DEFINITION 3.1 For any $k \in \mathbf{Z}_+$, a matrix A is said to be an $\mathbf{INS}_k$-matrix, $A \in \mathbf{INS}_k$, if and only if

$$A \in \bigcup_n \{\, M \in \Re^{n\times n} : |\mathrm{sol}(q, M)| = k, \text{ for all } q \in \mathrm{int}\ \mathrm{K}(M) \,\}.$$

DEFINITION 3.2 A matrix A is said to be an **INS**-matrix, $A \in \mathbf{INS}$ (*I*nvariant *N*umber of *S*olutions), if and only if

$$A \in \bigcup_{k \in \mathbb{Z}_+} \mathbf{INS}_k .$$

As before, and as will be shown in Theorem 4.7, we must define these classes with respect to q in the *interior* of $\mathrm{K}(M)$, not all of $\mathrm{K}(M)$, otherwise these classes will contain only the **P**-matrices. Notice that we have

$$\mathbf{U} = \mathbf{INS}_1 \subseteq \mathbf{INS} .$$

Thus the **INS**-matrices seem a natural extention of the **U**-matrices, but are strictly larger as seen by

EXAMPLE 3.3 Let

$$M = \begin{bmatrix} 0 & -2 \\ -2 & 1 \end{bmatrix}.$$

As illustrated in Figure 3.1, $M \in \mathbf{INS}_2$. Notice that the full complementary cones can be partitioned into two groups, $\{ C(\{1,2\}) \}$ and $\{ C(\emptyset), C(\{2\}) \}$, such that the union of the cones in each group covers the interior of $\mathrm{K}(M)$, and the interiors of the cones in each group are pairwise disjoint. We also see that $|\mathrm{sol}(q, M)|$ for $q \in \partial\,\mathrm{K}(M)$ is one or infinity – never two – for points in, respectively, $\mathrm{pos}\, C(\overline{n})_{\cdot 2}$ and $\mathrm{int\,pos}\, C(\overline{n})_{\cdot 1}$.

In the last chapter we noticed that $\mathbf{U} \cap \mathbf{Q} = \mathbf{P}$. A result of Murty's shows that a similar result holds for the class **INS**.

THEOREM 3.4 $\mathbf{INS} \cap \mathbf{Q} = \mathbf{P}$.

Proof. If $M \in \mathbf{INS} \cap \mathbf{Q}$, then $\mathrm{int}\,\mathrm{K}(M) = \Re^n$, so $|\mathrm{sol}(q, M)|$ is constant for all $q \in \Re^n$. Theorem 7.10 from Murty (1972) states that this constant is

equal to one. Hence $M \in \mathbf{U}$, and we have $M \in \mathbf{P}$ as desired. $\square$

Before continuing on to the next sections, where we look at what goes into making an INS-matrix, there are a few concepts which should be brought up first.

DEFINITION 3.5 Let $M \in \Re^{n \times n}$, we then define

$$K(M) = \bigcup_{\substack{\alpha \in (\overline{n}) \\ i \in \overline{n}}} \operatorname{pos} C(\alpha)_{\cdot \hat{i}} .$$

$K(M)$ is the union of the faces of the complementary cones. It contains, in some cases equals, $\partial \mathrm{K}(M)$. In Example 3.3, $(1,0)^T \in K(M) \setminus \partial \mathrm{K}(M)$, while with $M = 0$ we have $K(M) = \partial \mathrm{K}(M)$. $K(M)$ is the set of all $q \in \Re^n$ that are degenerate with respect to M. Being the union of a finite collection of sets with dimension $n-1$ or less, $K(M)$ has zero n-dimensional volume. It is a closed cone in $\Re^n$.

We will be interested in the open set $\Re^n \setminus K(M)$. Let Σ be the collection of the connected components of $\Re^n \setminus K(M)$. As $\Re^n$ is locally path connected and as $\Re^n \setminus K(M)$ is open, the path components of $\Re^n \setminus K(M)$ are the same as the (connected) components. See, for example, Munkres (1975). Σ "almost" partitions $\Re^n$, in that it partitions $\Re^n \setminus K(M)$ which is "almost" $\Re^n$. If $\Gamma \in \Sigma$, then Γ is an open polyhedral cone, i.e., $\partial \overline{\Gamma}$ is a finite collection of $(n-1)$-dimensional finite cones. It is *not* necessarily true that $\Gamma = \operatorname{int} \overline{\Gamma}$, although it will be shown later that $\Gamma \subseteq \operatorname{int} \overline{\Gamma}$. For example

EXAMPLE 3.6 Let

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

then Σ contains two components: $\Gamma_1 = \operatorname{int} \Re^3_+$ and

$$\Gamma_2 = \operatorname{int}[\Re^3 \setminus \Re^3_+] \setminus \{ x \in \Re^3 : x_1 = 0, x_3 \geq 0 \} .$$

$\Gamma_2 \neq \operatorname{int} \overline{\Gamma}_2 = \operatorname{int}[\Re^3 \setminus \Re^3_+]$. This also shows that, if $\Gamma \in \Sigma$, then Γ is *not* necessarily convex. For another example of this

EXAMPLE 3.7 Let

$$M = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

Then Σ contains three components:

$$\begin{aligned} \Gamma_1 &= \operatorname{int}[\Re^3 \setminus \Re^3_+], \\ \Gamma_2 &= \operatorname{int} \operatorname{pos} C(\{3\}) = \{ x \in \Re^3 : x_1 > x_3,\ x_2 > x_3,\ x_3 > 0 \}, \\ \Gamma_3 &= \operatorname{int} \Re^3_+ \setminus \operatorname{pos} C(\{3\}). \end{aligned}$$

Here only Γ_2 is convex, although $\Gamma_i = \operatorname{int} \overline{\Gamma}_i$ for $i = 1, 2, 3$. We will return later to the subject of convexity and the Γ_i.

We now discuss necessary conditions for a matrix to be INS.

3.2 Necessary Conditions for INS-matrices

In the last section we introduced the partition of $\Re^n \setminus K(M)$ by open polyhedral cones $\Gamma \in \Sigma$. The importance of this structure is contained in

THEOREM 3.8 If $\Gamma \in \Sigma$, and $q, \tilde{q} \in \Gamma$, then

$$|\operatorname{sol}(q, M)| = |\operatorname{sol}(\tilde{q}, M)|.$$

Proof. Fix $q, \tilde{q} \in \Gamma \in \Sigma$. As $q, \tilde{q} \notin \mathcal{K}(M)$, we know that q and $\tilde{q}$ are not contained in any degenerate complementary cone, and are not contained in the boundary of any nondegenerate cone. From Chapter 1, we know that any solution to (q, M) is associated with a complementary cone containing q. We also know that, if the cone is nondegenerate, there is only one solution associated with it. Now if $q \in \operatorname{pos} C(\alpha)$ then $q \in \operatorname{int} \operatorname{pos} C(\alpha)$. Letting $x = C(\alpha)^{-1} q > 0$, the solution associated with this cone is (w, z), where $z_\alpha = x_\alpha > 0$ and $w_{\bar{\alpha}} = x_{\bar{\alpha}} > 0$. As in Lemma 2.7, any other solution $(\tilde{w}, \tilde{z})$ is associated with another complementary cone $\operatorname{pos} C(\beta)$ containing q. Also, any other complementary cone containing q is associated with a different solution. We therefore see that $|\operatorname{sol}(q, M)|$ is the number of complementary cones that contains q. The same holds for $\tilde{q}$.

Suppose that for some $\alpha \in (\overline{n})$ we have $q \in \operatorname{pos} C(\alpha)$ and $\tilde{q} \notin \operatorname{pos} C(\alpha)$. Then any path from q to $\tilde{q}$ must contain a point in $\partial \operatorname{pos} C(\alpha) \subseteq \mathcal{K}(M)$, so q and $\tilde{q}$ are not in the same path component of $\Re^n \setminus \mathcal{K}(M)$, i.e., not in the same Γ, a contradiction. Thus any complementary cone containing q contains $\tilde{q}$, and vice versa. Thus they are in the same number of complementary cones, and so $|\operatorname{sol}(q, M)| = |\operatorname{sol}(\tilde{q}, M)|$.

$\square$

The proof just given shows that for any complementary cone, $\operatorname{pos} C(\alpha)$, and any $\Gamma \in \Sigma$,

$$\Gamma \cap \operatorname{pos} C(\alpha) \neq \emptyset \quad \Leftrightarrow \quad \Gamma \subseteq \operatorname{pos} C(\alpha).$$

The main result of this section is

THEOREM 3.9 If $M \in \mathbf{INS}$, then

$$\partial \mathrm{K}(M) = \bigcup_{\substack{\alpha \in (\overline{n}) \\ i \in \overline{n}}} \{ \operatorname{pos} C(\alpha)_{\cdot i} : \operatorname{pos} C(\alpha)_{\cdot i} \text{ is } \textit{not} \text{ proper} \} .$$

Proof. Let $\operatorname{pos} C(\alpha)$ be a degenerate cone. Suppose $\operatorname{pos} C(\alpha) \cap \operatorname{int} \mathrm{K}(M) \neq \emptyset$. Then there will exist a q such that

$$q \in \operatorname{int} \operatorname{pos} C(\alpha) \cap \operatorname{int} \mathrm{K}(M) .$$

From Proposition 1.6, we know that $|\mathrm{sol}(q, M)| = \infty$. As $M \in \mathbf{INS}$, there must be infinitely many solutions for each point in the interior of $\mathrm{K}(M)$. From the proof of Theorem 3.8, we see that for any point in $\Re^n \setminus \mathcal{K}(M)$ the number of solutions it has to the LCP is equal to the number of complementary cones containing it, which is finite. Hence $\operatorname{int} \mathrm{K}(M) \subseteq \mathcal{K}(M)$, but this is impossible as the set on the left is n-dimensional and the set on the right is $(n-1)$-dimensional. Thus all degenerate cones are contained in $\partial \mathrm{K}(M)$. (This also shows that $\mathbf{INS}_\infty = \emptyset$, so our definitions cover just what we want without any technical problems.)

Suppose now that $\operatorname{pos} C(\alpha)$ is a full cone, $\operatorname{pos} C(\alpha)_{\cdot i}$ is a reflecting face, and $\operatorname{pos} C(\alpha)_{\cdot i} \cap \operatorname{int} \mathrm{K}(M) \neq \emptyset$. Then there is a $q \in \operatorname{int} \operatorname{pos} C(\alpha)_{\cdot i} \cap \operatorname{int} \mathrm{K}(M)$ such that for any $\beta \in (\overline{n})$, $j \in \overline{n}$, we have

$$q \in \operatorname{pos} C(\beta)_{\cdot j} \quad \Rightarrow \quad \begin{cases} \dim[\operatorname{pos} C(\beta)_{\cdot j}] = n - 1 \\ q \in \operatorname{int} \operatorname{pos} C(\beta)_{\cdot j} \subseteq \operatorname{span} C(\alpha)_{\cdot i} \end{cases} \tag{3.1}$$

and any small enough open ball around q is bisected by $\operatorname{int} \operatorname{pos} C(\alpha)_{\cdot i}$ with the two open half-balls contained in $\Gamma_0, \Gamma_1 \in \Sigma$, respectively. (We are not

assuming $\Gamma_0 \neq \Gamma_1$.) Refer to Figure 3.2. To see this more clearly, notice that the set of points that are in either

(i) the boundary of an $(n-1)$-dimensional face of a complementary cone,

(ii) a k-dimensional complementary cone where $k < n-1$,

(iii) the intersection of $\operatorname{pos} C(\alpha)_{\cdot \bar{i}}$ with an $(n-1)$-dimensional face (of a complementary cone) *not* in $\operatorname{span} C(\alpha)_{\cdot \bar{i}}$,

is a set of dimension less than $n-1$, while $\dim[\operatorname{int} \operatorname{pos} C(\alpha)_{\cdot \bar{i}}] = n-1$. Furthermore, as all the k-dimensional facets of $\mathrm{K}(M)$ are closed and finite in number, we know that for an open ball around q, that has a small enough radius, we will have a k-dimensional facet of $\mathrm{K}(M)$ intersecting the open ball if and only if that facet contains q.

Since $\operatorname{pos} C(\alpha)_{\cdot \bar{i}}$ is a face of the full complementary cone $\operatorname{pos} C(\alpha)$, then either $\Gamma_0 \cap \operatorname{pos} C(\alpha) \neq \emptyset$ or $\Gamma_1 \cap \operatorname{pos} C(\alpha) \neq \emptyset$, but not both as $\operatorname{pos} C(\alpha)$ lies entirely on one side of $\operatorname{pos} C(\alpha)_{\cdot \bar{i}}$. Thus without loss of generality we assume

$$\Gamma_0 \subseteq \operatorname{pos} C(\alpha) \quad \text{and} \quad \Gamma_1 \cap \operatorname{pos} C(\alpha) = \emptyset .$$

(Thus, indeed, $\Gamma_0 \neq \Gamma_1$.) Let H_0 and H_1 be the two closed half-spaces with $\operatorname{span} C(\alpha)_{\cdot \bar{i}}$ as boundary, where $\Gamma_0 \subseteq H_0$ and $\Gamma_1 \subseteq H_1$. Suppose that there is some complementary cone, $\operatorname{pos} C(\beta)$, that contains Γ_1 but not Γ_0. Then it must be a full cone and have some face, say $\operatorname{pos} C(\beta)_{\cdot \bar{j}}$, containing q. By (3.1) this face lies in $\operatorname{span} C(\alpha)_{\cdot \bar{i}}$, hence $C(\beta)_{\cdot i}$ lies in H_1. However, as $\operatorname{pos} C(\alpha)_{\cdot \bar{i}}$ is reflecting we have both $I_{\cdot i}$ and $-M_{\cdot i}$ in $\operatorname{int} H_0$, a contradiction. Thus no complementary cone contains Γ_1 and not Γ_0. But $\operatorname{pos} C(\alpha)$ contains Γ_0 and not Γ_1. Hence,

$$|\mathrm{sol}(q^1, M)| \leq |\mathrm{sol}(q^0, M)| + 1 \qquad q^0 \in \Gamma_0,\ q^1 \in \Gamma_1 . \tag{3.2}$$

Since $q \in \operatorname{int} \mathrm{K}(M)$, we have $|\mathrm{sol}(q^1, M)| > 0$, so $\Gamma_0 \cup \Gamma_1 \subseteq \operatorname{int} \mathrm{K}(M)$. Hence (3.2) implies that $M \notin \mathbf{INS}$, a contradiction. Thus

$$M \in \mathbf{INS} \quad \Rightarrow \quad \partial \mathrm{K}(M) \supseteq \bigcup_{\substack{\alpha \in (\bar{n}) \\ i \in \bar{n}}} \{ \operatorname{pos} C(\alpha)_{\cdot \bar{i}} : \operatorname{pos} C(\alpha)_{\cdot \bar{i}} \text{ is } \textit{not} \text{ proper} \}.$$

Now suppose that $q \in \partial \mathrm{K}(M)$. Clearly q is not interior to any full cone. Suppose that it is not contained in a degenerate cone. Then it is on the boundary of some full cone, hence $q \in \partial \operatorname{int} \mathrm{K}(M)$. As $\operatorname{int} \mathrm{K}(M)$ is an n-dimensional polyhedral cone, $\partial \operatorname{int} \mathrm{K}(M)$ is the union of finitely many $(n-1)$-dimensional finite cones, each contained in some degenerate cone or a face of a full cone. If $\operatorname{pos} C(\alpha)_{\cdot \bar{i}}$ is a proper face, then we know $I_{\cdot i}$ and $-M_{\cdot i}$ are on opposite sides of $\operatorname{span} C(\alpha)_{\cdot \bar{i}}$. Thus

$$\operatorname{int} \operatorname{pos} C(\alpha)_{\cdot \bar{i}} \subseteq \operatorname{int}[\operatorname{pos} C(\alpha) \cup \operatorname{pos} C(\alpha \triangle \{ i \})] \subseteq \operatorname{int} \mathrm{K}(M),$$

giving

$$\dim[\operatorname{pos} C(\alpha)_{\cdot \bar{i}} \cap \partial \mathrm{K}(M)] < n - 1.$$

Thus $\operatorname{pos} C(\alpha)_{\cdot \bar{i}}$ is not a face containing one of the $(n-1)$-dimensional finite cones of $\partial \operatorname{int} \mathrm{K}(M)$. Thus $\partial \operatorname{int} \mathrm{K}(M)$ is contained in the reflecting faces and the degenerate cones, and, hence, so is $\partial \mathrm{K}(M)$.

$\square$

Corollary 3.10 Let $M \in \Re^{n \times n}$, then

$$\partial \mathrm{K}(M) \subseteq \bigcup_{\substack{\alpha \in (\bar{n}) \\ i \in \bar{n}}} \{ \operatorname{pos} C(\alpha)_{\cdot \bar{i}} : \operatorname{pos} C(\alpha)_{\cdot \bar{i}} \text{ is } \textit{not} \text{ proper} \}.$$

Proof. Simply notice that in the last part of the proof of Theorem 3.9 we never used the fact that $M \in \mathrm{INS}$ when showing this result. $\square$

Saigal (1972b) uses the concept of a "regular pseudomanifold." We borrow the terminology for the similar, but stronger, concept embodied in

DEFINITION 3.11 Let $M \in \Re^{n\times n}$, then $\mathrm{K}(M)$ is said to be *regular* if and only if

$$\partial\,\mathrm{K}(M) = \bigcup_{\substack{\alpha\in(\overline{n})\\ i\in\overline{n}}} \{\,\mathrm{pos}\,C(\alpha)_{\cdot\hat{i}} : \mathrm{pos}\,C(\alpha)_{\cdot\hat{i}} \text{ is } \textit{not} \text{ proper}\,\}\,.$$

Theorem 3.9 then says that

$$M \in \mathrm{INS} \quad \Rightarrow \quad \mathrm{K}(M) \text{ is regular.}$$

This is the general necessary condition for a matrix to be in INS. In the next section we take up the question of this condition's sufficiency.

3.3 Sufficient Conditions for INS-matrices

We now know that if a matrix M is in INS then $\mathrm{K}(M)$ is regular. The natural question is to ask whether this is a sufficient condition. To this end, we prove the

LEMMA 3.12 Assume $M \in \Re^{n\times n}$ and $\mathrm{K}(M)$ is regular. Assume also that $\Gamma_0, \Gamma_1 \in \Sigma$ are subsets of $\mathrm{K}(M)$ — and hence its interior. Suppose for some $x \in \Gamma_0$ and $y \in \Gamma_1$ there is a path $L \in \mathrm{int}\,\mathrm{K}(M)$ from x to y. Then

L can be chosen to have the following "nondegeneracy" properties:

(i) $L \cap K(M)$ is a finite set;

(ii) if q is a point in $L \cap K(M)$,
then q is in the interior of any face containing it;

(iii) all faces containing q lie in the same hyperplane.

Proof. If $\Gamma_0 = \Gamma_1$, by definition, we can construct a path L^* within Γ_0 from x to y. The above are then vacuously true. If $\Gamma_0 \neq \Gamma_1$, we can construct the path L^* from L as follows. We know that L is the image of some continuous function

$$f : [0,1] \to \Re^n, \qquad f(0) = x \in \Gamma_0, \qquad f(1) = y \in \Gamma_1.$$

Since $\overline{\Gamma}_0$ is closed, we have

$$0 < \lambda = \max\{ f^{-1}(\overline{\Gamma}_0) \} < 1.$$

Let $q = f(\lambda)$. Then $q \in \partial\overline{\Gamma}_0$. Let B be an open ball in int $\mathrm{K}(M)$ around $q \in L \subseteq$ int $\mathrm{K}(M)$. Since all the facets are closed sets, we may assume that B is so small that any facet of $\mathrm{K}(M)$, of any dimension, intersecting B must contain q. See Figure 3.3 for a picture of the local situation around q.

Γ_0 is a component so we may construct a path L^* from x to q where $L^* \setminus q \subseteq \Gamma_0$. Let $\overline{q} \in B \cap (L^* \setminus q)$. We claim that for each point in $B \setminus K(M)$ there is a path in B from that point to $\overline{q}$ that satisfies the conditions of the lemma. Clearly, if such a path exists between $\overline{q}$ and some point in $B \cap \Gamma_i$, then one exists between $\overline{q}$ and all points in $B \cap \Gamma_i$. (This does not follow from what has been set up as it could be that $B \cap \Gamma_i$ is not connected. In

this case we may temporarily take the Γ_i as the connected components of $B \setminus K(M)$ and all will go through. It will turn out, in the next chapter, that this precaution is not necessary. However, we do need to know that the path can be built within B for later reference.) The set of points in B that can be connected to $\bar{q}$ by a path satisfying the given conditions is, then, the closure of the union of some of the $B \cap \Gamma_i$. Call this set S. S is the intersection of B with a polyhedral cone with vertex translated to q. It is n-dimensional as $B \cap \Gamma_0 \subseteq S$. If $S \neq B$, then S has a boundary in B. We may then find a point $\tilde{q} \in B$, in the interior of one of the $(n-1)$-dimensional faces making up ∂S, such that the faces of $\mathrm{K}(M)$ containing $\tilde{q}$ all lie on the same hyperplane and all contain $\tilde{q}$ in their interiors. (These restrictions will remove a set of points that is $(n-2)$-dimensional at most, and we have a set that is $(n-1)$-dimensional from which to choose.) A sufficiently small line segment, $\tilde{L}$, with $\tilde{q}$ as midpoint and orthogonal to the (unique) boundary face of S through $\tilde{q}$, will make a path from some $r_0 \in \operatorname{int} S$ to some $r_1 \in B \setminus S$ where

$$\tilde{L} \cap K(M) = \tilde{q}.$$

The conditions of the lemma are satisfied for this path. Since $r_0 \in S$, we have a path to r_0 from $\bar{q}$ satisfying the conditions. Combining the paths gives a path from $\bar{q}$ to $r_1 \in B \setminus S$ satisfying the conditions, a contradiction. Thus $B = S$.

Now, let

$$\lambda' = \max\{ f^{-1}(\bar{\Gamma}_i) : B \cap \Gamma_i \neq \emptyset \}.$$

Clearly $\lambda < \lambda'$, as L did not end at q. Let

$$q' = f(\lambda') \in \bar{\Gamma}_2.$$

Pick a point r in $B \cap \Gamma_2$. There will exist a path, from x to $\overline{q}$, from $\overline{q}$ to r, and from r to q', satisfying the conditions of the lemma. If $\lambda' = 1$, then $q' = y$ and we are done. If $\lambda' < 1$, an open ball around q' can be made small enough to repeat the above arguments, extending the path into some new component Γ_3. As there are finitely many components, we will eventually have a path from x to y in int $\mathrm{K}(M)$ that satisfies the lemma's conditions.

$\square$

Consider a point $q \in L \cap K(M)$. The previous lemma shows that for a small enough open ball, B, around q, there is a hyperplane H such that $q \in B \cap K(M) = B \cap H$, and $B \cap H$ splits B into two open hemi-hyperspheres, contained in, say, Γ_2 and Γ_3 respectively. (See Figure 3.4.) Since $q \in$ int $\mathrm{K}(M)$, all faces containing q are proper. Suppose that a full complementary cone, $\operatorname{pos} C(\alpha)$, contains Γ_2 but not Γ_3. Hence for some $i \in \overline{n}$, we must have $q \in \operatorname{pos} C(\alpha)_{\cdot i} \subseteq H$. The previous lemma allows us to assume that $\operatorname{int} \operatorname{pos} C(\alpha)_{\cdot i}$ bisects B into the aforementioned hemi-hyperspheres. As $\operatorname{pos} C(\alpha)_{\cdot i}$ is proper, $I_{\cdot i}$ and $-M_{\cdot i}$ lie on opposite sides of H. Thus $\operatorname{pos} C(\alpha \triangle \{i\})$ contains Γ_3 but not Γ_2. Since we could have assumed at the start that $\operatorname{pos} C(\alpha)$ contained Γ_3 and not Γ_2, we have a bijective correspondence between complementary cones containing Γ_2, not Γ_3, and complementary cones containing Γ_3, not Γ_2. So the number of complementary cones containing Γ_2 is the same as the number containing Γ_3. Thus

$$q \in \Gamma_2,\ \tilde{q} \in \Gamma_3 \quad \Rightarrow \quad |\mathrm{sol}(q, M)| = |\mathrm{sol}(\tilde{q}, M)|.$$

Therefore, if we start at x and follow the path L, we will pass through a finite sequence of $\Gamma_i \in \Sigma$ where $|\mathrm{sol}(q, M)|$ is invariant for all q in the Γ_i. Hence

$$|\mathrm{sol}(x, M)| = |\mathrm{sol}(y, M)| .$$

We have been assuming that $x, y \notin K(M)$. Now suppose we have $y \in \mathrm{int}\,\mathrm{K}(M) \cap K(M)$. As in the proof of Lemma 3.12, we can find an open ball $B \subseteq \mathrm{int}\,\mathrm{K}(M)$, with $y \in B$, so small that $B \cap K(M)$ is the intersection of B with the union of finitely many $(n-1)$-dimensional finite cones with vertex translated to y. (See Figure 3.5.) Since y is contained in only full complementary cones, each cone containing y is associated with exactly one solution in $\mathrm{sol}(y, M)$. Suppose that $\Gamma_1 \subseteq \mathrm{pos}\, C(\alpha)$. Then $y \in \mathrm{pos}\, C(\alpha)$ and let the associated solution be (w, z). We will show that no other cone containing Γ_1 has (w, z) as the associated solution to (y, M).

We may assume that $\alpha = \emptyset$ as we can always block pivot on $M_{\alpha\alpha}$ to get the principal transform $\overline{M}$. As shown in Chapter 1, the cone structure is preserved and working with $\mathrm{pos}\, C_{\overline{M}}(\emptyset)$ is equivalent to working with $\mathrm{pos}\, C_M(\alpha)$. Thus $(w, z) = (y, 0)$. We may assume that $\mathrm{supp}\, w = \mathrm{supp}\, y = \overline{n} \setminus \overline{k}$, where $0 < k \leq n$. Thus a full cone, $\mathrm{pos}\, C(\beta)$, has $(y, 0)$ as its associated solution to (y, M) if and only if $\beta \in (\overline{k})$. However, for all $\beta \in (\overline{k})$ and for all $i \in \overline{k}$, we have $y \in \mathrm{pos}\, C(\beta)_{\cdot i}$. Hence $\mathrm{pos}\, C(\beta)_{\cdot i}$ is a proper face of $\mathrm{K}(M)$ as $y \in \mathrm{int}\,\mathrm{K}(M)$. Therefore, if $\beta, \gamma \in (\overline{k})$, then $(\det M_{\beta\beta})(\det M_{\gamma\gamma}) > 0$. As $\emptyset \in (\overline{k})$, we then see that $M_{\overline{kk}} \in \mathbf{P}$. If $\beta \in (\overline{k})$, and there is some $0 < \tilde{y} \in \mathrm{pos}\, C(\beta)$, then $\mathrm{pos}\, C(\beta)_{\overline{kk}}$ is associated with a solution to $(\tilde{y}_{\overline{k}}, M_{\overline{kk}})$. As $M_{\overline{kk}} \in \mathbf{P}$ and $\tilde{y}_{\overline{k}} > 0$, there is only one such solution and it is associated with only the positive orthant. Thus $\beta = \emptyset$. Yet $\Gamma_1 \subseteq \mathrm{int}\,\mathrm{pos}\, C(\emptyset) = \mathrm{int}\, \mathfrak{R}^n_+$. We may conclude, as claimed, that $\mathrm{pos}\, C(\alpha)$ is the only complementary cone containing Γ_1 with (w, z) as the associated solution to $(y, 0)$. Hence $|\mathrm{sol}(y, M)|$ is at least as large as the number of complementary cones containing Γ_1.

Suppose now that $y \in \operatorname{pos} C(\alpha)$. Thus looking at Figure 3.5 again we have that some Γ_k containing y is contained in $\operatorname{pos} C(\alpha)$. In fact, $B \cap \operatorname{pos} C(\alpha)$ is the closure of the union of sets in the form $B \cap \Gamma_i$. Select two points $q \in \Gamma_1$ and $\tilde{q} \in \Gamma_k$. Let L be a path in B between q and $\tilde{q}$ satisfying the conditions of Lemma 3.12. (In the proof we noted that such a path can be made within B.) Suppose we cross a boundary of $\operatorname{pos} C(\alpha)$ moving along L from $\tilde{q}$ to q. We will leave the cone at some point interior to a face, say $\operatorname{pos} C(\alpha)_{\cdot\bar{i}}$. This face must be proper, as it contains a point in $\operatorname{int} \mathrm{K}(M)$. We then have that $I_{\cdot i}$ and $-M_{\cdot i}$ lie on opposite sides of $\operatorname{span} C(\alpha)_{\cdot\bar{i}}$. Let $\beta = \alpha \triangle \{i\}$. Thus we enter $\operatorname{pos} C(\beta)$ when we leave $\operatorname{pos} C(\alpha)$. Moreover, $L \subseteq B$ so $\operatorname{pos} C(\alpha)_{\cdot\bar{i}} \cap B \neq \emptyset$ implying $y \in \operatorname{pos} C(\alpha)_{\cdot\bar{i}} = \operatorname{pos} C(\beta)_{\cdot\bar{i}}$. Hence the solutions to (y, M) associated with both $\operatorname{pos} C(\alpha)$ and $\operatorname{pos} C(\beta)$ are the same, both using only the columns in $\operatorname{pos} C(\alpha)_{\cdot\bar{i}}$. Thus we will eventually reach a full cone containing Γ_1 such that the solution to (y, M) it is associated with and the solution to (y, M) that $\operatorname{pos} C(\alpha)$ is associated with are the same. Hence $|\mathrm{sol}(y, M)|$ equals the number of complementary cones containing Γ_1. We have thus shown

THEOREM 3.13 Let $M \in \Re^{n \times n}$. If $\mathrm{K}(M)$ is regular, and S is a connected component of $\operatorname{int} \mathrm{K}(M)$, then

$$q, \tilde{q} \in S \quad \Rightarrow \quad |\mathrm{sol}(q, M)| = |\mathrm{sol}(\tilde{q}, M)|.$$

□

We get the partial converse to Theorem 3.9

COROLLARY 3.14 Let $M \in \Re^{n \times n}$. If $\mathrm{K}(M)$ is regular, and $\operatorname{int} \mathrm{K}(M)$ is connected, then $M \in \mathbf{INS}$.

□

Example 2.3 shows an INS-matrix for which int $K(M)$ is disconnected. In fact, there exist points in $K(M)$, for example $(1,1)^T$ and $(-1,-1)^T$, which can be connected in $K(M)$ only with paths containing the origin. Since $K(M)$ is a cone, any two of its points can be connected by a path through the origin, so this particular $K(M)$ is just "barely" connected. However, we note

THEOREM 3.15 Let $M \in \Re^{n \times n}$, $n > 1$. If no complementary cone is strongly degenerate, then any two nonzero points in $K(M)$ can be connected by a path in $K(M)$ not containing the origin.

Proof. Define the map $F : \Re^n \to \Re^n$ as

$$F(x) = \sum_{i=1}^{n} \left(\max(x_i, 0) \cdot I_{\cdot i} + \min(x_i, 0) \cdot M_{\cdot i} \right).$$

Thus $K(M) = F(\Re^n)$. Clearly F is continuous. Define the continuous radial projection $P : \Re^n \setminus \{0\} \to S^{n-1}$ as $P(x) = x/\|x\|$, where

$$S^{n-1} = \{ x \in \Re^n : \|x\| = 1 \}$$

is the unit sphere in n-space. Since no complementary cone is strongly degenerate, $F(x) = 0$ implies that $x = 0$. So $0 \notin F(S^{n-1})$, hence $P \circ F : S^{n-1} \to S^{n-1}$ is a continuous mapping. Furthermore

$$P \circ F(S^{n-1}) = S^{n-1} \cap K(M).$$

This and the path connectedness of S^{n-1} imply that $S^{n-1} \cap K(M)$ is path connected. However, any nonzero point in $K(M)$ can be connected by a

path to $S^{n-1} \cap \mathrm{K}(M)$, i.e., the ray through that point from the origin. The theorem follows.

$\square$

EXAMPLE 3.16 The matrix

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix}$$

belongs to INS_2. However, int $\mathrm{K}(M)$ is not connected, and no complementary cone is strongly degenerate. This example shows that we cannot have a result similar to the previous one concerning the connectedness of int $\mathrm{K}(M)$ in the weakly degeneracy case. However, in the case of nondegeneracy we have

THEOREM 3.17 Let $M \in \Re^{n \times n}$. If no complementary cone is degenerate, then int $\mathrm{K}(M)$ is connected.

Proof. Take $q, \tilde{q} \in \operatorname{int} \mathrm{K}(M)$. We can find full complementary cones so that $q \in \operatorname{pos} C(\alpha)$ and $\tilde{q} \in \operatorname{pos} C(\beta)$. If $\alpha = \beta$, then q and $\tilde{q}$ are path connected within $\operatorname{int} \operatorname{pos} C(\alpha) \subseteq \operatorname{int} \mathrm{K}(M)$, even though q and $\tilde{q}$ may be the only points of the path not in $\operatorname{int} \operatorname{pos} C(\alpha)$.

Suppose $\alpha \vartriangle \beta = \{i\}$. If $\operatorname{pos} C(\alpha)_{\cdot \bar{i}}$ is reflecting, then $I_{\cdot i}$ and $-M_{\cdot i}$ lie on the same side of $\operatorname{span} C(\alpha)_{\cdot \bar{i}}$. By Lemma 2.8, $\operatorname{int} \operatorname{pos} C(\alpha) \cap \operatorname{int} \operatorname{pos} C(\beta) \neq \emptyset$. We can thus build a path in $\operatorname{int} \operatorname{pos} C(\alpha)$ from q to a point in this intersection, and then to $\tilde{q}$ through $\operatorname{int} \operatorname{pos} C(\beta)$. If $\operatorname{pos} C(\alpha)_{\cdot \bar{i}}$ is proper, then $I_{\cdot i}$ and $-M_{\cdot i}$ lie on opposite sides of $\operatorname{span} C(\alpha)_{\cdot \bar{i}}$. So

$$\operatorname{int} \operatorname{pos} C(\alpha)_{\cdot \bar{i}} \subseteq \operatorname{int}[\operatorname{pos} C(\alpha) \cup \operatorname{pos} C(\beta)] \subseteq \operatorname{int} \mathrm{K}(M) .$$

The path can then be constructed from q though $\operatorname{int}\operatorname{pos} C(\alpha)$ to a point within $\operatorname{int}\operatorname{pos} C(\alpha)_{\cdot i}$, and from there, through $\operatorname{int}\operatorname{pos} C(\beta)$, to $\tilde{q}$.

In general, if $\alpha \triangle \beta = \{ i_1, \ldots, i_k \}$, let $\gamma_1 = \alpha$ and for $1 \leq j \leq k$ let $\gamma_{j+1} = \gamma_j \triangle \{ i_j \}$. Then, $\operatorname{pos} C(\gamma_j)$ and $\operatorname{pos} C(\gamma_{j+1})$ are adjacent for $1 \leq j \leq k$; moreover, $\gamma_{k+1} = \beta$. This and the previous arguments show that $\operatorname{int} \mathrm{K}(M)$ will contain a path from q to $\tilde{q}$. That is, $\operatorname{int} \mathrm{K}(M)$ is path connected.

□

We conclude this chapter with a partial characterization of the class **INS**.

COROLLARY 3.18 Let $M \in \Re^{n \times n}$, and suppose that M has no zero principal minors. We then have

$$M \in \mathbf{INS} \quad \Leftrightarrow \quad \mathrm{K}(M) \text{ is regular.}$$

□

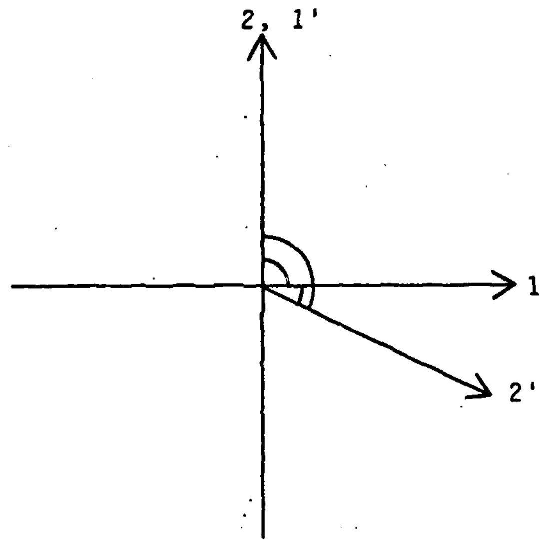

Figure 3.1

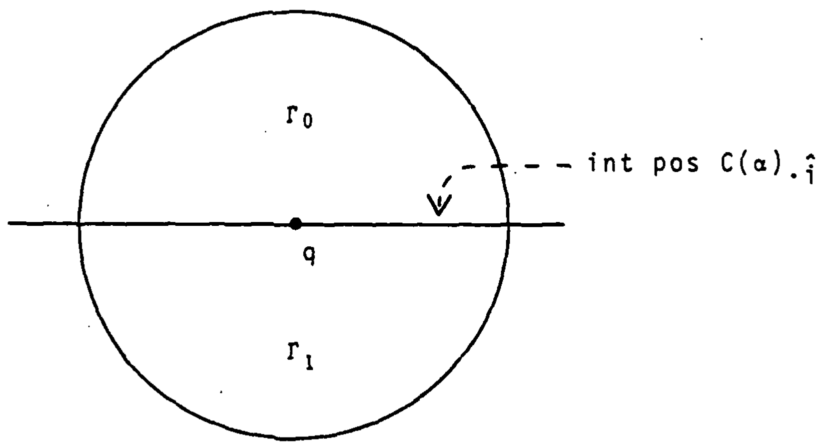

Figure 3.2

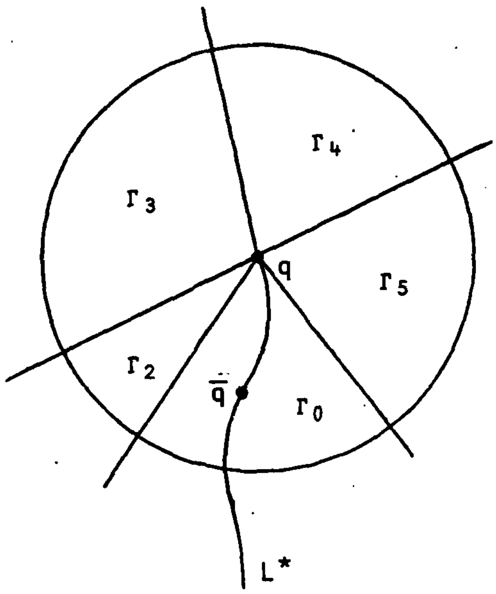

Figure 3.3

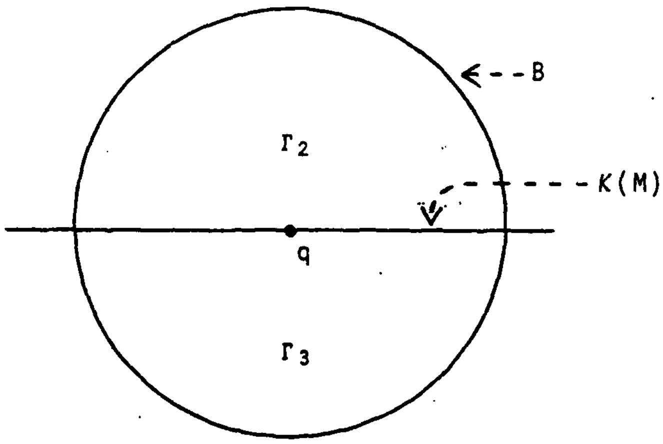

Figure 3.4

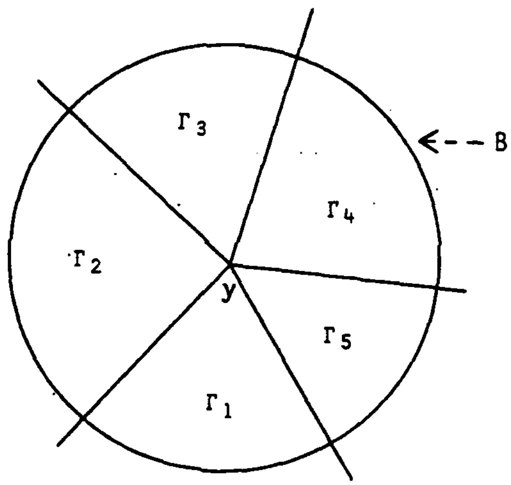

Figure 3.5

CHAPTER 4.
INS-MATRICES: FURTHER RESULTS

4.1 Convexity of the Γ

The partition Σ, defined in the last chapter, was seen to be an important object. We noted in Example 3.7 that a component $\Gamma \in \Sigma$ need not be convex, even if $\Gamma \in \mathrm{K}(M)$. The matrix used in the example was a degenerate matrix, but degeneracy was unneccessary as the matrix

$$M = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

is nondegenerate and has, geometrically, the same Σ as the matrix in Example 3.7. However, we do have the result

THEOREM 4.1 If $M \in \mathrm{INS} \cap \Re^{n \times n}$, then all $\Gamma \in \Sigma$ contained in $\mathrm{K}(M)$ are convex.

Before starting the proof, we will need the following lemma.

LEMMA 4.2 If $M \in \Re^{n\times n}$ and $\Gamma \neq \operatorname{int}\overline{\Gamma}$ for some $\Gamma \in \Sigma$, then $\Gamma \subseteq \operatorname{int}\overline{\Gamma}$ and $x \in \operatorname{int}\overline{\Gamma} \setminus \Gamma$ implies that x is in a degenerate cone.

Proof. Γ is open and contained in $\overline{\Gamma}$. As $\operatorname{int}\overline{\Gamma}$ is the largest open set in $\overline{\Gamma}$, it follows that $\Gamma \subseteq \operatorname{int}\overline{\Gamma}$.

Now $x \in \overline{\Gamma} \setminus \Gamma = \partial\Gamma \subseteq \mathcal{K}(M)$. Thus x is contained in the boundary of some complementary cone, say $\operatorname{pos} C(\alpha)$. Suppose $\operatorname{pos} C(\alpha)$ is a *full* cone. Then either $\Gamma \subseteq \operatorname{pos} C(\alpha)$, or $\Gamma \cap \operatorname{pos} C(\alpha) = \emptyset$.

In the first case, $x \in \partial[\Re^n \setminus \operatorname{pos} C(\alpha)] \subseteq \overline{\Re^n \setminus \operatorname{pos} C(\alpha)}$. Notice $\overline{\Gamma} \subseteq \operatorname{pos} C(\alpha)$ as $\operatorname{pos} C(\alpha)$ is closed. Hence $\overline{\Re^n \setminus \operatorname{pos} C(\alpha)} \cap \operatorname{int}\overline{\Gamma} = \emptyset$, a contradiction.

In the second case, $x \in \partial \operatorname{pos} C(\alpha)$. As $\operatorname{pos} C(\alpha)$ is full, for all $\epsilon > 0$, the set $B(x,\epsilon) \cap \operatorname{pos} C(\alpha)$ is n-dimensional. Now $\partial\Gamma \subseteq \mathcal{K}(M)$ is $(n-1)$-dimensional at most, so $B(x,\epsilon) \setminus \overline{\Gamma} \neq \emptyset$ for all $\epsilon > 0$. Thus $x \in \overline{\Re^n \setminus \overline{\Gamma}}$, and so $x \notin \operatorname{int}\overline{\Gamma}$, a contradiction.

We have shown that $\operatorname{pos} C(\alpha)$ is a degenerate cone, as required. □

Proof of Theorem 4.1. Suppose there exists a nonconvex $\Gamma \in \mathrm{K}(M)$. Then there exist two points $x, y \in \Gamma$ such that the line segment between them,

$$L = \{\lambda x + (1-\lambda)y : 0 \leq \lambda \leq 1\},$$

is not contained in Γ. Thus there must exist a point $q \in L \cap \partial\Gamma \cap \mathcal{K}(M)$. We may assume

$$q = L \cap \mathcal{K}(M) \cap B(q,\epsilon), \tag{4.1}$$

for some small $\epsilon > 0$. To see this, notice that, for small $\epsilon > 0$,

$K(M) \cap B(q,\epsilon)$ is the intersection of $B(q,\epsilon)$ with a finite collection of finite cones with vertex q and dimension less than or equal to $n-1$. Since Γ is open, we may take $\epsilon_x, \epsilon_y > 0$ small enough so that

$$B_x = B(x, \epsilon_x) \subseteq \Gamma \quad \text{and} \quad B_y = B(y, \epsilon_y) \subseteq \Gamma .$$

We may thus take x to be any point in B_x and y to be any point in B_y. This means we may "perturb" x and y, and hence the line segment L, with n-dimensional "freedom." We can thus perturb L so that it contains q and satisfies (4.1). See Figure 4.1.

For the moment assume that $q \in \operatorname{int} \mathrm{K}(M)$. Thus q is not in any degenerate cone, so we know from the previous lemma that $q \in \overline{\Gamma} \cap \overline{\operatorname{int}[\Re^n \setminus \Gamma]}$. Thus, for all $\epsilon > 0$, $K(M) \cap B(q,\epsilon)$ must be $(n-1)$-dimensional. Since we can perturb L with n-dimensional freedom, we may assume that for q, and some $\epsilon > 0$ small enough, $K(M) \cap B(q,\epsilon) = H \cap B(q,\epsilon)$ for some hyperplane H, see Figure 4.2, and that any face of any complementary cone containing q is $(n-1)$-dimensional and contains q in its interior. (The argument here is similar to several given before. We are selecting from a set that is $(n-1)$-dimensional and eliminating a set that is at most $(n-2)$-dimensional.) Now let $\operatorname{pos} C(\alpha)_{\cdot i}$ be a face containing q. $q \in \operatorname{int} \mathrm{K}(M)$ implies that this is a proper face, so as $q \in \partial\Gamma$ we may assume $\Gamma \subseteq \operatorname{pos} C(\alpha)$, for otherwise $\Gamma \subseteq \operatorname{pos} C(\alpha \vartriangle \{i\})$. But $\operatorname{pos} C(\alpha)_{\cdot i} \subseteq H$, so $\operatorname{pos} C(\alpha)$, and hence Γ, lies entirely on one side of H. But L crosses H with $x \in \Gamma$ on one side and $y \in \Gamma$ on the other. Contradiction.

Now assume $q \in \partial \mathrm{K}(M)$. This implies $q \in \partial \operatorname{int} \mathrm{K}(M)$. Again, by the perturbation argument given above and the fact that $\partial \operatorname{int} \mathrm{K}(M)$ is a finite set of $(n-1)$-dimensional finite cones, we can assume q is contained in the interior of some face $\operatorname{pos} C(\alpha)_{\cdot i}$ of which L is a transversal. As q is in $\partial\Gamma$

and in $\partial\,\mathrm{int}\,\mathrm{K}(M)$, there is some full cone, say $\mathrm{pos}\,C(\alpha)$, that has a face in the affine hull of $\mathrm{pos}\,C(\alpha)_{\cdot i}$ and contains Γ. Thus Γ is, again, only on one side of the affine hull of $\mathrm{pos}\,C(\alpha)_{\cdot i}$. Contradiction.

□

As a side result, Lemma 4.2 implies

COROLLARY 4.3 If $M \in \Re^{n\times n}$ is nondegenerate, then for all $\Gamma \in \Sigma$ we have $\Gamma = \mathrm{int}\,\overline{\Gamma}$.

□

We remark that, even for nondegenerate $M \in \mathrm{INS}$, if $\Gamma \not\subseteq \mathrm{K}(M)$ then Γ may not be convex. For example, in $\Re^2$ if we let $M = -I \in \mathrm{INS}_4$ then we get $|\Sigma| = 2$ where one component is $\mathrm{int}\,\mathrm{K}(M) = \mathrm{int}\,\Re^2_+$ and convex, with the other component being $\Re^2 \setminus \Re^2_+$ and nonconvex.

Failing to show for nondegenerate M that all the $\Gamma \in \Sigma$ are convex, one might consider showing that some particular Γ is convex. With this in mind, we prove the next theorem before leaving this section. Recall that, by Theorem 3.8, the number $|\mathrm{sol}(q, M)|$ is invariant over $q \in \Gamma$ for each $\Gamma \in \Sigma$.

THEOREM 4.4 Let $M \in \Re^{n\times n}$ be nondegenerate. There exists at least one $\Gamma^* \in \Sigma$ such that for all $\Gamma \in \Sigma$

$$|\mathrm{sol}(q^*, M)| \geq |\mathrm{sol}(q, M)|, \qquad \text{for } q^* \in \Gamma^*,\ q \in \Gamma, \tag{4.2}$$

and any such Γ^* must be convex.

Proof. It is clear that at least one Γ^* exists. As in the proof for Theorem 4.1, we assume otherwise, and take $x, y \in \Gamma^*$ such that the line segment between them, L, contains a point q not in Γ^*. As before, using the nondegeneracy of M, we may assume $q \in \partial\Gamma^*$ and that there is a hyperplane H, of which

L is a transversal, such that if q is in any face of any complementary cone, then q is in the interior of the face, the face is $(n-1)$-dimensional, and the face is contained in H. We also know at least one face of a complementary cone contains q. If a complementary cone, with a face containing q, contains Γ^*, then, as in the proof of Theorem 4.1, we will have Γ^* lying entirely on one side of H. As before, this contradicts the fact that L is a transversal of H. Thus no complementary cone with a face contained in H can contain Γ^*. By nondegeneracy and the fact that some face does contain q and hence is in H, we know some full complementary cone does have a face lying in H. That cone must contain Γ, where Γ is the other component in Σ that has q on its boundary. (Since for $\epsilon > 0$ small enough we know that $B(q,\epsilon) \setminus K(M)$ is two hemi-hyperspheres, one on each side of H, we see that at most two components in Σ contain q on their boundaries. We know $q \in \partial\Gamma^*$ and we have just seen that another component must also have q on its boundary.) Hence, every complementary cone containing Γ^* also contains Γ, but some cone containing Γ does not contain Γ^*. Thus, with $q \in \Gamma$ and $q^* \in \Gamma^*$, we have $|\mathrm{sol}(q, M)| > |\mathrm{sol}(q^*, M)|$. This contradicts (4.2).

□

4.2 The Number of Solutions

In discussing the class **INS** an important question to ask is for what values of k is $\mathbf{INS}_k$ empty? We know $\mathbf{INS}_1 = \mathbf{U}$ is certainly nonempty. It can be easily seen that for all positive integers n, $-I \in \Re^{n \times n}$ is in $\mathbf{INS}_{2^n}$. What about values of k other than the powers of two? We will attempt to give evidence suggesting that $\mathbf{INS}_k = \emptyset$ if k is *not* a power of two. We begin

by proving

THEOREM 4.5 Suppose $M \in \mathrm{INS}_k \cap \Re^{n \times n}$. If there exists some point in $\partial\,\mathrm{K}(M)$ that is *not* contained in a strongly degenerate cone, then k is even.

Proof. Let $q \in \partial\,\mathrm{K}(M)$ be contained only in full or weakly degenerate cones. By dimensional arguments similar to ones given previously, we may assume there is a hyperplane H such that if q is contained in a face of a complementary cone, then that face is $(n-1)$-dimensional with q in its interior, and the face is contained in H. We can then take an $\epsilon > 0$ so small that $B(q,\epsilon) \cap \mathrm{int}\,\mathrm{K}(M) \subseteq \Gamma$ for some particular $\Gamma \in \Sigma$. See Figure 4.3. Any full complementary cone containing q must contain Γ, and likewise any complementary cone containing Γ must contain q. Since there are no strongly degenerate cones containing q, by Lemma 3.2 of Saigal (1972a) it follows that q is contained in an even number of full cones. Thus for any $\tilde{q} \in \Gamma$, we have $|\mathrm{sol}(\tilde{q}, M)|$ is even, whence k is even.

□

COROLLARY 4.6 Suppose $M \in \mathrm{INS}_k \cap \Re^{n \times n}$. If there are no strongly degenerate cones in $\mathrm{K}(M)$, then k is even, or $M \in \mathbf{P}$.

□

We now reconsider the proof of Theorem 4.5. This time we will allow strongly degenerate cones. If q is contained in a degenerate face, then $|\mathrm{sol}(q, M)| = \infty$. Otherwise q is contained only in reflecting faces, as $q \in \partial\,\mathrm{K}(M)$. Thus q is contained only in full cones. Let $(w, z) \in \mathrm{sol}(q, M)$ be the solution associated with a full cone $\mathrm{pos}\, C(\alpha)$ that contains q, and so there is an $i \in \bar{n}$ such that $q \in \mathrm{int}\,\mathrm{pos}\, C(\alpha)_{\cdot i}$. Thus $z_{\alpha \setminus \{i\}} > 0$,

$w_{\bar{\alpha}\setminus\{i\}} > 0$ and $z_i = w_i = 0$. Hence, (w, z) is also the solution associated with the full cone $\operatorname{pos} C(\alpha \vartriangle \{i\})$, and is associated with *no* other full cone. Thus, as we had q contained in k full cones, it follows that $|\mathrm{sol}(q, M)| = \frac{k}{2}$. In any case, $|\mathrm{sol}(q, M)| \neq k$. This reasoning, along with Theorem 3.4, proves the following assertion which was mentioned at the beginning of Chapter 3.

THEOREM 4.7

$$\mathbf{P} = \bigcup_{k \in \mathbf{Z}_+} \left\{ \bigcup_{n \in \mathbf{Z}_+} \{ M \in \Re^{n \times n} : |\mathrm{sol}(q, M)| = k, \text{ for all } q \in \mathrm{K}(M) \} \right\}$$

$\square$.

At the start of this section it was suggested that $\mathrm{INS}_k = \emptyset$ if k is not a power of two. As will be shown later, this would follow from

CONJECTURE 4.8 Let $M \in \mathrm{INS}_k \cap \Re^{n \times n}$. If $\mathrm{K}(M)$ has *no* reflecting faces, then $k \leq 2$.

The author has examined many INS-matrices, and studied their general structure in the case where all boundary faces are degenerate. No counterexample to Conjecture 4.8 has been found. To obtain some feeling for why the conjecture should be true, let us consider trying to construct $\mathrm{K}(M)$ for an INS_k matrix, $k \geq 3$, with all boundary faces degenerate. Clearly $\partial\,\mathrm{int}\,\mathrm{K}(M) \neq \emptyset$, otherwise $M \in \mathbf{P}$. Let H be a hyperplane, let $C = H \cap \partial\,\mathrm{int}\,\mathrm{K}(M)$, and suppose that $\dim C = n-1$. Since only degenerate faces are in $\partial\,\mathrm{K}(M)$, each such face acts as the "base" of at most one full complementary cone. We would then find that every point in C that is not in a m-dimensional facet of a complementary cone, where $m \leq n-2$, i.e., "almost all" the points in C, must be contained in exactly k degenerate faces

which act as bases for k full complementary cones. In building $\mathrm{K}(M)$ we find there is a "tradeoff" in our placement of the column vectors of $[\,I \mid -M\,]$. The more we place in C, the more degenerate faces we will have to form bases of full cones, which can be used for this multiple covering of C. The more we place outside of C, the more full cones we can actually form on these degenerate faces. There are other tradeoffs in the construction. For instance, the more boundary hyperplanes H that $\partial\,\mathrm{int}\,\mathrm{K}(M)$ has, i.e., the more possible C's that exist, the more we must worry about putting the column vectors of $[\,I \mid -M\,]$ on the boundary of each C to "spread them around" to the different C's. The fewer the number of boundary hyperplanes, the more likely the C's will contain lower dimensional linear spaces (linealities), requiring many degenerate faces for our multiple coverings of the C's, and the previously mentioned tradeoff becomes more critical. With these and other requirements on the structure of $\mathrm{K}(M)$, including the way in which the columns vectors of $[\,I \mid -M\,]$ form the complementary cones, it seems certain that the $2n$ column vectors of $[\,I \mid -M\,]$ would not permit k to exceed two. *If this is so,* we have

THEOREM 4.9 If $M \in \mathbf{INS}_k \cap \Re^{n\times n}$ and Conjecture 4.8 is true, then k is a power of two.

Proof. The proof uses induction on n. If $n = 1$, then there are at most two complementary cones. Thus $k \leq 2$ and the theorem is true. Suppose the theorem is true for $n-1$. If no faces are reflecting, then $k \leq 2$ by Conjecture 4.8 and the theorem holds.

Thus suppose $\mathrm{pos}\,C(\alpha)_{\cdot\bar{\imath}}$ is a reflecting face. Then, $\mathrm{pos}\,C(\alpha)_{\cdot\bar{\imath}} \subseteq \partial\,\mathrm{int}\,\mathrm{K}(M)$. Let H be the hyperplane $\mathrm{span}\,C(\alpha)_{\cdot\bar{\imath}}$. Let

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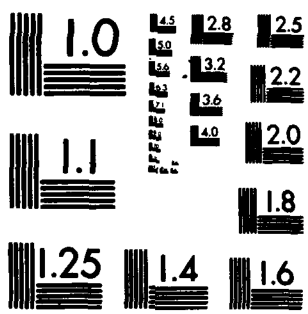

MICROCOPY RESOLUTION TEST CHART
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$$S = \bigcup_{\substack{\beta \in (\bar{\pi}) \\ i \in \bar{\pi}}} \{ \operatorname{pos} C(\beta)_{\cdot\hat{\imath}} : \operatorname{pos} C(\beta)_{\cdot\hat{\imath}} \subseteq H \}.$$

$\operatorname{pos} C(\beta)_{\cdot\hat{\imath}}$ is in S if and only if the columns of $C(\beta)_{\cdot\hat{\imath}}$ are all in S. (Notice that we use $\cdot\hat{\imath}$ here as both $I_{\cdot i}$ and $-M_{\cdot i}$ are on the same side of H, but *not* in H.) We can now think of the vectors of $I_{\cdot\hat{\imath}}$ and $-M_{\cdot\hat{\imath}}$ as forming an $(n-1)$-dimensional LCP. (For notation, say that the matrix associated with this new LCP is $\tilde{M}$.) The correspondence is as follows:

H takes the place of $\Re^{n-1}$;

S takes the place of $\mathrm{K}(\tilde{M})$;

$\operatorname{pos} C(\alpha)_{\cdot\hat{\imath}}$ takes the place of the identity matrix as $\operatorname{pos} C(\alpha)_{\cdot\hat{\imath}}$ is a known *full* cone in H;

if $\operatorname{pos} C(\hat{\alpha})_{\cdot j} \in H$, then $-\tilde{M}_{\cdot j}$ is represented by $\operatorname{pos} C(\hat{\alpha})_{\cdot j}$, otherwise $-\tilde{M}_{\cdot j}$ is represented by the zero vector. (Here we index on $\hat{\imath}$, so we have $j \in \hat{\imath}$.)

We will refer to this LCP in H as the *reduced* LCP. We claim that $\mathrm{K}(\tilde{M})$ is regular.

Suppose $q \in \operatorname{int} S$ is contained in a reflecting face of the reduced LCP. By dimensional arguments similar to earlier ones, we may assume there is an $(n-2)$-dimensional hyperplane $\tilde{H} \subseteq H$ such that if a face of a complementary cone in the reduced LCP contains q, that face is $(n-2)$-dimensional, contains q in its interior, and is contained in $\tilde{H}$. Thus for $\epsilon > 0$ small enough, $B(q,\epsilon) \cap H$ is a hypersphere divided into two hemi-hyperspheres by $\tilde{H}$, with one hemi-hypersphere contained in $\partial\Gamma$ and the other contained in $\partial\Gamma'$. Here Γ and Γ' are in the Σ of the *original* LCP. See Figure 4.4. Let $\operatorname{pos} C(\alpha)_{\cdot\hat{\imath}\hat{\jmath}}$ be a reflecting face in S containing q. Thus

both full cones of the reduced LCP which contain that face lie on the same side of $\tilde{H}$ in H. We may assume they both contain $\partial\Gamma \cap H$, and both intersect $\partial\Gamma' \cap H$ only on $\tilde{H}$. But then, as both $I_{\cdot j}$ and $-M_{\cdot j}$ lie in H on the same side as $\partial\Gamma \cap H$, no full cone of S can contain $\partial\Gamma' \cap H$ and without containing $\partial\Gamma \cap H$. Thus more cones of the reduced LCP contain $\partial\Gamma \cap H$ than contain $\partial\Gamma' \cap H$. But each full cone of the reduced LCP is a face for exactly two cones of the original LCP. (The cones you get by adding in $I_{\cdot i}$ and $-M_{\cdot i}$, respectively, as another generator of the cone.) Also, each full cone of the original LCP with a face in H has that face as a full cone of the reduced LCP. Hence as $q \in \operatorname{int} S$, we have some full cone of S containing $\partial\Gamma' \cap H$ and so if $x \in \Gamma$ and $y \in \Gamma'$ then $|\operatorname{sol}(x, M)| > |\operatorname{sol}(y, M)| > 0$. This contradicts the assumption that $M \in \mathbf{INS}$.

Now suppose $q \in \operatorname{int} S$ is contained in a degenerate cone, say $\operatorname{pos} C_{\tilde{M}}(\alpha)$, of the reduced LCP, where $i \notin \alpha \in (\overline{n})$. If all the columns in $C_{\tilde{M}}(\alpha)$ are from the original LCP, i.e., none of them are zero columns made, as mentioned before, because the associated $-M_{\cdot j}$ was not in H, then $\operatorname{pos}[\, I_{\cdot i} \mid C_{\tilde{M}}(\alpha)\,]$ is a degenerate cone of the original LCP. What's more, as $q \in \operatorname{int} S$, this degenerate cone contains points in the interior of the convex hull of S and $I_{\cdot i}$, which, in turn, is contained in $\mathrm{K}(M)$. This is impossible since $M \in \mathbf{INS}$. We thus assume $C_{\tilde{M}}(\alpha)$ contains columns which were made zero, as described before. Now substitute for all but one of these columns that were made zero, say all but $-\tilde{M}_{\cdot j}$, the associated complementary column from $C_{\tilde{M}}(\emptyset)$. Let this new matrix be $C_{\tilde{M}}(\beta)$, and notice that $q \in \operatorname{pos} C_{\tilde{M}}(\beta)$. If $C_{\tilde{M}}(\beta \triangle \{j\})$ is a degenerate cone in the reduced LCP, then as none of its columns were made zero in the way initially described, we would be back to the previous case. Thus assume that $\operatorname{pos} C_{\tilde{M}}(\beta \triangle \{j\})$ is a full cone in the reduced LCP, and thus $\dim[\operatorname{pos} C_{\tilde{M}}(\beta)_{\cdot \bar{j}}] = n - 2$. We can now use the same argument as

in the case when we assumed q was in a reflecting face of S. However, here we use $\operatorname{pos} C_{\tilde{M}}(\beta)_{\cdot\hat{\jmath}}$ instead of $\operatorname{pos} C(\alpha)_{\cdot\hat{\imath}\hat{\jmath}}$. Also, there is one full cone, not two, of the reduced LCP with $\operatorname{pos} C_{\tilde{M}}(\beta)_{\cdot\hat{\jmath}}$ as a face, but we still have this one cone containing $\partial\Gamma \cap H$ – and not containing $\partial\Gamma' \cap H$ – so the argument remains valid. We conclude that $S = \mathrm{K}(\tilde{M})$ is regular, as claimed.

Let q be in a connected component of the interior of S. By familiar dimensional arguments, we may assume that if a complementary cone of the reduced LCP contains q, then it is a full cone containing q in its interior. Thus, for an $\epsilon > 0$ small enough, $B(q,\epsilon) \cap \operatorname{int} \mathrm{K}(M) \subseteq \Gamma$ for some particular $\Gamma \in \Sigma$. (See Figure 4.3 again.) Since $q \in \partial \mathrm{K}(M)$, the complementary cones of the reduced LCP that contain q are reflecting faces of the original LCP. (They can't be degenerate faces as both $I_{\cdot i}$ and $-M_{\cdot i}$ are not in H.) Thus each cone of the reduced problem that contains q is the face of two distinct full cones of the original problem, and these two cones will contain Γ. Also, any cone containing Γ must contain q. As we've seen before, the number of cones containing Γ must be k, hence the number of full cones in the reduced LCP containing q must be $\frac{k}{2}$. Since q could be in any connected component of $\operatorname{int} S$, using Theorem 3.13 we find $\tilde{M} \in \mathrm{INS}_{k/2}$. By induction on the dimension of the LCP we see that $\frac{k}{2}$ is a power of two. Thus k is a power of two.

□

The previous theorem makes it seem almost certain that

$$\mathrm{INS} = \bigcup_{p=0}^{\infty} \mathrm{INS}_{2^p}.$$

However, there is a large class of matrices for which we can show the result of the theorem holds without recourse to Conjecture 4.8. We see this in the

following

THEOREM 4.10 Let $M \in \mathbf{INS}_k \cap \Re^{n \times n}$. Suppose that for all $\alpha \in (\overline{n})$ we have $\det C(\alpha) = 0$ if and only if $C(\alpha)_{\cdot i} = 0$ for some $i \in \overline{n}$. It is then the case that k is a power of two.

Proof. The point here is to show that the proof of Theorem 4.9 goes through – without using Conjecture 4.8 and with only minor changes – when we restrict ourselves to the matrices described in the statement of Theorem 4.10. (We use here the notation of the proof of Theorem 4.9.)

In the case where we have a reflecting face, the proof is the same. The only thing needing commentary is the induction step where we must now show the reduced LCP satisfies the hypothesis of this theorem. Suppose $\operatorname{pos} C_{\tilde{M}}(\alpha)$ is a degenerate cone in S, where $i \notin \alpha \in (\overline{n})$. Assume no column of $C_{\tilde{M}}(\alpha)$ is zero. Thus all the columns in $C_{\tilde{M}}(\alpha)$ come from the original LCP, i.e., are not "artifical" zero columns as described before, and so $\operatorname{pos} C(\alpha)$ is a degenerate cone of the original LCP with no zero columns. This contradicts the fact that the original problem satisfied the hypothesis of the theorem. Hence the reduced problem satisfies the hypothesis of the theorem.

Now suppose there are no reflecting faces. If $M \in \mathbf{Q}$, then $M \in \mathbf{P}$ and we're done. Otherwise $\partial \operatorname{int} \mathrm{K}(M) \neq \emptyset$ and so must be made up of degenerate cones. Thus M must have at least one column that is all zeros, say $M_{\cdot i} = 0$. Thus

$$\{ q \in \Re^n_+ : q_i = 0 \} \subseteq \partial \operatorname{int} \mathrm{K}(M),$$

and we can let

$$H = \{ q \in \Re^n : q_i = 0 \}.$$

We can now go through with the proof of Theorem 4.9 for the case of a reflecting face. The reduced LCP is made in the same way. $I_{.\hat{\imath}}$ represents $C_{\tilde{M}}(\emptyset)$ taking the place of the identity matrix for the reduced LCP. For $j \in \hat{\imath}$, if $-M_{.j} \in H$ then $-M_{.j}$ represents $-\tilde{M}_{.j}$, otherwise $-\tilde{M}_{.j} = 0$. The difference is that each full cone in S is the face of *one* full cone in the original LCP – which will contain $I_{.i}$ – and for each full cone of the original LCP with a face in H, that face will be a full cone in S. We will finally get the reduced LCP in INS_k, which, by induction, will mean k is a power of two. (The reduced LCP satisfies the hypothesis of this theorem by the same reasoning as given in the second paragraph of this proof.) We thus arrive at the same conclusion as in Theorem 4.9.

□

We leave this section with the following immediate corollary to the last theorem.

COROLLARY 4.11 *If $M \in \text{INS}_k \cap \Re^{n\times n}$ is nondegenerate, then k is a power of two.*

□

4.3 The Structure of $\mathrm{K}(M)$ and $\partial\mathrm{K}(M)$

The purpose of this section is to build a link between the combinatorial and geometric representations of $\mathrm{K}(M)$ for nondegenerate INS-matrices. The main result is to show $\mathrm{K}(M)$ and $\partial\mathrm{K}(M)$ can be divided into several disjoint pseudomanifolds. For this purpose we review some of the basic definitions related to pseudomanifolds. For a more detailed discussion of

these topological-combinatorial constructs discussion see, for example, Eaves (1972, 1976), Freund (1980), and Spanier (1966).

DEFINITION 4.12 Let V be a finite, non-empty set of elements (vertices). We say that a collection P of subsets of V is an *n-dimensional pseudomanifold* if and only if

(i) $S \in P$ implies that $|S| = n + 1$. The subsets S are referred to as *n-simplexes.*

(ii) $F \subseteq V$ and $|F| = n$ implies that F is a subset of at most two elements in P. (F is an $(n-1)$*-simplex.*)

(iii) For every pair $S, \tilde{S} \in P$, there is a finite sequence $S = S_0, S_1, \ldots, S_m = \tilde{S}$ of elements of P such that $|S_i \cap S_{i+1}| = n$, for $0 \leq i < m$.

The *boundary,* ∂P, of the pseudomanifold P is the collection of subsets $F \subseteq V$ which have n elements and are contained in exactly one element of P.

DEFINITION 4.13 Let S be a simplex of the n-dimensional pseudomanifold P. Let $(s_0, s_1, \ldots, s_n)$ be some fixed ordering of the elements of S. Any ordering of these elements, say $(s_{j_0}, s_{j_1}, \ldots, s_{j_n})$, is then defined to be a *positive (negative) orientation* if and only if the permutation $(j_0, j_1, \ldots, j_n)$ is even (odd). In this way we say we have *oriented* the simplex S. We say two distinct simplexes in P are *adjacent* if they have n elements in common. Thus, if S and $\tilde{S}$ are adjacent, we can write $S = (s, s_1, \ldots, s_n)$ and $\tilde{S} = (\tilde{s}, s_1, \ldots, s_n)$. If these particular orderings for S and $\tilde{S}$ are given different signs by the orientations on S and $\tilde{S}$, then we say S and $\tilde{S}$ are *coherently oriented.* Finally, we say P is *orientable* if we can specify an orientation for all $S \in P$ such that any two adjacent simplexes are coherently

oriented.

EXAMPLE 4.14 For any matrix $M \in \Re^{n \times n}$, $\mathrm{K}(M)$ can be viewed as the geometric representation of an orientable $(n-1)$-dimensional pseudomanifold without boundary, i.e., the boundary is an empty set. (Notice the combinatorial dimension is one less than the geometric dimension.) Let V be the set of column vectors in the matrix $[\ I \mid -M\ ]$. The elements of the pseudomanifold are the sets of column vectors of the complementary matrices. The geometric representation of $C(\alpha)_{\cdot\beta}$ is then $\operatorname{pos} C(\alpha)_{\cdot\beta}$ for $\alpha, \beta \in (\overline{n})$. For any $\alpha \in (\overline{n})$, let the orientation of $(C(\alpha)_{\cdot 1}, \ldots, C(\alpha)_{\cdot n})$ be determined by the sign of $(-1)^{|\alpha|}$. It is not hard to see this will orient the pseudomanifold.

Doverspike and Lemke (1981) showed that for a large class of nondegenerate matrices $M \in \mathbf{Q}_0$, it is possible to find a collection of complementary cones whose union is $\mathrm{K}(M)$, and forms a pseudomanifold P in such a way that the geometric union of the faces in ∂P is $\partial\,\mathrm{K}(M)$. Furthermore, there will be exactly one other collection, disjoint from the first, of complementary cones whose union is also $\mathrm{K}(M)$, which also is a pseudomanifold whose boundary is ∂P. As we will be building somewhat similar pseudomanifolds from INS-matrices, eventually to prove Theorem 4.18 – which the reader may wish to glance at now – it will be useful at this point to go over the proof of the Doverspike-Lemke result before proceeding. The basic idea of the proof is explained in the following paragraph. (The full details of the proof would require many pages and are omitted.)

Consider the geometric structure of $\mathrm{K}(M)$. For each 1-dimensional facet of $\mathrm{K}(M)$ we find a column from $[\ I \mid -M\ ]$ whose "pos" spans it. (We have our choice of any column vector which is in the facet when there is more than

one.) In this way we build up (trivial) pseudomanifolds for the 1-dimensional facets of $\mathrm{K}(M)$. From here on, we assume we have built up pseudomanifolds for the r-dimensional facets of $\mathrm{K}(M)$, $1 \leq r < n$. The boundary of any $(r+1)$-dimensional facet is the union of r-dimensional facets. We can take the union of the pseudomanifolds of these r-dimensional facets as a boundaryless pseudomanifold over the geometric boundary of our selected $(r+1)$-dimensional facet. (They will "fit" together as their boundaries were made from the same pseudomanifolds on the $(r-1)$-dimensional facets.) We then give a construction to show there will be exactly two pseudomanifolds, as previously described, on the $(r+1)$-dimensional facet whose boundary pseudomanifold is the pseudomanifold we pieced together on the geometric boundary of the $(r+1)$-dimensional facet. We continue this until $r+1=n$, at which point we have the result.

The concept we wish to use from this is the family of pseudomanifolds on the r-dimensional facets of $\mathrm{K}(M)$, with the r-dimensional pseudomanifolds forming the boundaries of the $(r+1)$-dimensional pseudomanifolds. However, we will be working from the higher dimensions to the lower dimensions, whereas Doverspike and Lemke do the opposite. Notice we are able to start our constructions since $\mathrm{K}(M)$ is regular, which implies that for any face in $\mathrm{K}(M)$, say $\operatorname{pos} C(\alpha)_{\cdot\bar{\imath}}$, we have

$$\dim[\operatorname{pos} C(\alpha)_{\cdot\bar{\imath}} \cap \partial \mathrm{K}(M)] = n-1 \quad \Rightarrow \quad \operatorname{pos} C(\alpha)_{\cdot\bar{\imath}} \subseteq \partial \mathrm{K}(M).$$

The following lemma will prove useful.

LEMMA 4.15 Let $M \in \mathbf{INS} \cap \Re^{n\times n}$ be nondegenerate. It is then the case that the r-dimensional facets of $\mathrm{K}(M)$ are regular. (That is, if the $(r-1)$-dimensional cone $\operatorname{pos} C(\alpha)_{\cdot\beta}$, where $\alpha, \beta \in (\overline{n})$ and $|\beta| = r-1$, is the common face of two r-dimensional complementary cones in an r-dimensional

facet of $\mathrm{K}(M)$, then it is either a boundary face of the facet, or it is a proper face. If the two r-dimensional cones are not in the same r-dimensional facet of $\mathrm{K}(M)$ then, clearly, $\operatorname{pos} C(\alpha)_{\cdot\beta}$ is on the common boundary of both r-dimensional facets which contain the r-dimensional cones, so we need not worry about this case.)

Proof. This is easily seen by reverse induction. It is true for dimension n, as $\mathrm{K}(M)$ is regular by assumption. Suppose it is true for dimension $r+1$, $1 \leq r < n$. Suppose that it fails in dimension r. We may assume some q in the interior of an r-dimensional facet is contained in a reflecting $(r-1)$-dimensional face, $\operatorname{pos} C(\alpha)_{\cdot\overline{r-1}}$, which is the common face of the two cones $\operatorname{pos}[\ C(\alpha)_{\cdot\overline{r-1}} \mid -M_{\cdot r}\ ]$ and $\operatorname{pos}[\ C(\alpha)_{\cdot\overline{r-1}} \mid I_{\cdot r}\ ]$ contained in the r-dimensional facet. (As M is nondegenerate, there cannot be any degenerate faces here.) Some $(r+1)$-dimensional facet will contain this r-dimensional facet in its boundary, and thus must contain some column vector from $[\ I \mid -M\ ]$ which is not in $[I_{\cdot\overline{r}}, -M_{\cdot\overline{r}}]$. Say it contains $I_{\cdot n}$. As the r-dimensional complementary cones covering the r-dimensional facet must be generated from column vectors of $[I_{\cdot\overline{r}}, -M_{\cdot\overline{r}}]$ – due to nondegeneracy of M – then the interior of the cone $\operatorname{pos}[\ q \mid I_{\cdot n}\ ]$ is contained in the interior of the $(r+1)$-dimensional facet. Hence the reflecting face, $\operatorname{pos}[\ C(\alpha)_{\cdot\overline{r-1}} \mid I_{\cdot n}\ ]$, which is the common face between the cones $\operatorname{pos}[\ C(\alpha)_{\cdot\overline{r-1}} \mid I_{\cdot r} \mid I_{\cdot n}\ ]$ and $\operatorname{pos}[\ C(\alpha)_{\cdot\overline{r-1}} \mid -M_{\cdot r} \mid I_{\cdot n}\ ]$, contains points in the interior of the $(r+1)$-dimensional facet, contradicting the regularity of that facet. This completes the induction. Thus all the r-dimensional facets are regular, for $1 \leq r \leq n$.

□

We can now start building up our pseudomanifolds.

DEFINITION 4.16 For any complementary cone, C, in $\mathrm{K}(M)$ define the pseudomanifold $P = P(C)$ to be C and all complementary cones C^* for which there exists a finite sequence of complementary cones $C = C_1, C_2, \ldots, C_m = C^*$, where, for $1 \leq i < m$, C_i and C_{i+1} are adjacent cones whose common face is proper.

Let $M \in \mathrm{INS}_k \cap \Re^{n \times n}$ be a given nondegenerate matrix. Thus by Theorem 3.17 we know that int $\mathrm{K}(M)$ is connected. Fix some $C \subseteq \mathrm{K}(M)$. Let $P = P(C)$. Let q and $\tilde{q}$ be two points in int $\mathrm{K}(M)$ such that if a complementary cone contains one of these points, then it contains that point in its interior, i.e., $q, \tilde{q} \in \mathrm{K}(M) \setminus \mathcal{K}(M)$. We can now use Lemma 3.12 to get a path L from q to $\tilde{q}$ satisfying the conditions of Lemma 3.12. Suppose s members of P contain q. Now move along L from q to $\tilde{q}$. When L crosses a face of a complementary cone, that face must be proper as $L \subseteq$ int $\mathrm{K}(M)$ and $\mathrm{K}(M)$ is regular. Thus L leaves one complementary cone and enter another one. If the first cone was a member of P, then the second cone will also be. Hence, for points in $L \setminus \mathcal{K}(M)$, the number of members of P that contain any given point is independent of the point selected. Thus $\tilde{q}$ is contained in s members of P, as was q. Thus every point in int $\mathrm{K}(M) \setminus \mathcal{K}(M)$ is contained in s members of P.

Before continuing on, let us digress momentarily to point out a simple fact about P. Suppose that C^* is a cone in P. By definition we have the sequence of cones, $C = C_1, C_2, \ldots, C_m = C^*$, adjacent on proper faces. Suppose $C_1 = \operatorname{pos} C(\alpha)$ and $C_2 = \operatorname{pos} C(\beta)$. By the definition of a proper face, we have $(\det M_{\alpha\alpha})(\det M_{\beta\beta}) > 0$. If we have $C^* = \operatorname{pos} C(\gamma)$, then it is easily seen that continuing the reasoning in the last sentence along the sequence of cones $C_1, C_2, \ldots, C_m$ will lead us to conclude $(\det M_{\alpha\alpha})(\det M_{\gamma\gamma}) > 0$. Hence, the sign of the determinant of the prin-

cipal submatrix of M associated with every member of P is invariant over P. This fact will be useful momentarily. For notational purposes, we say a full complementary cone $\mathrm{pos}\, C(\alpha)$ is *positive* (*negative*) if $\det M_{\alpha\alpha} > 0$ ($\det M_{\alpha\alpha} < 0$).

Returning to the main discussion, we have shown every point in $\mathrm{K}(M) \setminus K(M) = \mathrm{int}\ \mathrm{K}(M) \setminus K(M)$ is contained in s members of P, where, clearly, $s \geq 1$. Since these complementary cones are closed, and by non-degeneracy $\mathrm{K}(M) = \overline{\mathrm{int}\ \mathrm{K}(M) \setminus K(M)}$, it follows that the geometric union of the members of P is $\mathrm{K}(M)$.

Now, if $\mathrm{pos}\, C(\alpha)_{\cdot\bar{i}}$ is a face of exactly one member of P, i.e, is in ∂P, then it must be in $\partial\, \mathrm{K}(M)$. For if it contains a point in $\mathrm{int}\ \mathrm{K}(M)$, then it must be a proper face, which would imply either both or neither of the cones containing it are in P. Hence the geometric union of the members of ∂P must be contained in $\partial\, \mathrm{K}(M)$.

Let $q \in \partial\, \mathrm{K}(M)$ be such that if any complementary cone contains it, the cone contains it in the interior of one of its faces, and that face must be contained in $\partial\, \mathrm{K}(M)$. (This, as usual, allows q to be "almost all" the points in ∂S.) Hence, for $\epsilon > 0$ small enough, any point in

$$B(q, \epsilon) \cap \mathrm{int}\ \mathrm{K}(M) \subseteq \mathrm{K}(M) \setminus K(M) \tag{4.3}$$

will be in the same complementary cones as q. As the points in (4.3) are contained in s members of P, it follows that q is contained in s members of P. Suppose that $\mathrm{pos}\, C(\alpha) \in P$ contains q in its face $\mathrm{pos}\, C(\alpha)_{\cdot\bar{i}}$. Let $\beta = \alpha \vartriangle \{i\}$, thus $\mathrm{pos}\, C(\beta)$ is the one other complementary cone with $\mathrm{pos}\, C(\alpha)_{\cdot\bar{i}}$ as a face. The face cannot be proper as it is in $\partial\, \mathrm{K}(M)$. Thus the face is reflecting, and so $(\det M_{\alpha\alpha})(\det M_{\beta\beta}) < 0$. Since $\mathrm{pos}\, C(\alpha) \in P$, the

digression above shows that $\operatorname{pos} C(\beta) \notin P$. Hence, any face of a complementary cone in $\partial \mathrm{K}(M)$ is a face of at most one cone in P. Thus q is in s members of ∂P.

In the previous paragraph we showed that any cone $\operatorname{pos} C(\alpha)_{\cdot\beta}$, where $|\beta| = n-1$, is contained in at most one member of P. We now assume that this proper holds for lower dimensional cones.

ASSUMPTION 4.17 Let $M \in \mathrm{INS} \cap \Re^{n\times n}$ be nondegenerate, and let C be any complementary cone in $\mathrm{K}(M)$. Let F be any r-dimensional facet of $\mathrm{K}(M)$, $1 < r < n$, and $\operatorname{pos} C(\alpha)_{\cdot\beta} \subseteq \partial \mathrm{K}(M)$, be any cone in ∂F where $\alpha, \beta \in (\overline{n})$, $\operatorname{pos} C(\alpha) \in P(C)$, and $|\beta| = r-1$. Then there exists at most one $\gamma \in (\overline{n})$ with $|\gamma| = r$ such that, $\operatorname{pos} C(\alpha)_{\cdot\gamma} \subseteq F$ and $\beta \subsetneq \gamma$.

(This assumption is essentially the "consistency" assumption used in the previously cited work of Doverspike and Lemke.)

We can now state the main theorem of this section.

THEOREM 4.18 Let $M \in \mathrm{INS}_k \cap \Re^{n\times n}$ be nondegenerate. If Assumption 4.17 holds, then the complementary cones of $\mathrm{K}(M)$ can be partitioned into k disjoint collections where each collection is an orientable $(n-1)$-dimensional pseudomanifold by the representation described in Example 4.14. Furthermore if P is one of these pseudomanifolds, then the geometric union of the cones in P equals $\mathrm{K}(M)$. Also, if $\operatorname{pos} C(\alpha), \operatorname{pos} C(\beta) \in P$, then *int* $\operatorname{pos} C(\alpha) \cap$ *int* $\operatorname{pos} C(\beta) = \emptyset$. (In this way each pseudomanifold partitions $\mathrm{K}(M)$.) In addition, the $(n-1)$-faces making up the boundary of P, call it ∂P, also have disjoint interiors and their union is geometrically $\partial \mathrm{K}(M)$. (It is known that ∂P will be an orientable $(n-1)$-dimensional pseudomanifold without boundary.)

Proof. Most of the work has been already done. We will use reverse induction. Suppose by induction we have a sequence of facets of $K(M)$, say $F_r, F_{r+1}, \ldots, F_{n-1}$, and $\dim F_i = i$ for $r \le i < n$. In addition, suppose for each F_i there is a collection, P_i, of i-dimensional facets of members of $P = P(C)$, each facet being in F_i, such that for each $q \in F_i$ which is not contained in any $(i-1)$-dimensional facet of any complementary cone, (which is "almost all" of F_i,) there are exactly s members of P_i containing q.

We have already shown we may start the induction by taking F_{n-1} as any $(n-1)$-dimensional facet of $K(M)$, and selecting as P_{n-1} all those faces of members of P that lie in F_{n-1}. Now suppose we are at the general induction step. Take as F_{r-1} any boundary facet of F_r. For any $q \in F_{r-1}$ that is not contained in any $(r-2)$-dimensional facet of any complementary cone there is an $\epsilon > 0$ small enough so that each member of P_r either contains or is disjoint from $B(q,\epsilon) \cap F_r$. By induction, for small $\epsilon > 0$, we have each point in $B(q,\epsilon) \cap \operatorname{int} F_r$ must be in exactly s members of P_r, thus q is in s members of P_r. By Assumption 4.17, each $(r-1)$-dimensional facet, of a complementary cone, contained in F_{r-1} must be the face of no more than one member of P_r. Hence, if we define P_{r-1} as all the faces of members of P_r contained in F_{r-1}, the point q is contained in exactly s members of P_{r-1}. Noticing that the members of P_{r-1} must be facets of members of P completes the induction.

The only "catch" in the induction is where we assume that F_r has a boundary facet. Suppose it doesn't, and hence F_r is an r-dimensional subspace. By the nondegeneracy of M, we may assume $I_{.i}$ and $-M_{.i}$ are *not* in F_r, for $r < i \le n$. Also, notice F_r must be covered by the r-dimensional facets of complementary cones that are in F_r. If there is

some column vector of $[\, I_{\cdot\bar{r}} \mid -M_{\cdot\bar{r}} \,]$ that is *not* in F_r, say $I_{\cdot 1}$, then every r-dimensional facet, of a complementary cone, that *is* in F_r must contain $-M_{\cdot 1}$ and so by nondegeneracy must *not* contain $M_{\cdot 1}$. This contradicts the fact that F_r is covered. Thus F_r contains all the column vectors of $[\, I_{\cdot\bar{r}} \mid -M_{\cdot\bar{r}} \,]$. We can view F_r as $\Re^r$, and $[\, I_{\cdot\bar{r}} \mid -M_{\cdot\bar{r}} \,]$ as defining a LCP, where the matrix of the LCP is $M_r = -M_{\bar{r}\bar{r}}$. We know $M_r \in \mathbf{Q}$. By Lemma 4.15 $\mathrm{K}(M_r)$ is regular so, by Theorem 3.18, $M \in \mathbf{INS}$. Hence $M_r \in \mathbf{P}$. Thus $s = 1$.

Now suppose that we can always continue the induction down to F_1, no matter what choices we make along the way. By nondegenracy, F_1 can contain at most two column vectors from $[\, I \mid -M \,]$. If it contains only one such vector then we have $s = 1$ as before. If it contains two such vectors then they are $I_{\cdot i}$ and $-M_{\cdot i}$ for some $i \in \bar{n}$. (In this case $s = 2$.) Hence $-M_{\cdot i} = \lambda I_{\cdot i}$ for some $\lambda > 0$. As the boundary of $\mathrm{K}(M)$ contained no lineality (no linear subspace), there must be a minimum of n 1-dimensional facets. Hence, for some $i \in \bar{n}$, each such facet must contain $-M_{\cdot i}$ and $I_{\cdot i}$. Each facet must be associated with a different i. Thus M is a diagonal matrix with negative diagonal entries. It is easily seen that for each complementary cone C, we have $P(C) = \{C\}$. Hence $s = 1$, a contradiction.

In all cases we have have $s = 1$. Thus any two cones in P must have disjoint interiors, otherwise the intersection would be n-dimensional which would mean some of the points in the intersection are in $\mathrm{K}(M) \setminus \mathcal{K}(M)$ and would have to be in only one member of P. The same can be said for the members of ∂P. Since any point in $\mathrm{K}(M) \setminus \mathcal{K}(M)$ is contained in k complementary cones, as $M \in \mathbf{INS}_k$, and each complementary cone C is contained in some pseudomanifold P, for example $P(C)$, then the com-

plementary cones can be partitioned into k, clearly disjoint, collections with each collection forming a pseudomanifold. Each pseudomanifold, $P(C)$, can be oriented, as in Example 4.14, by giving the ordering $(C(\alpha)_{\cdot 1}, \ldots, C(\alpha)_{\cdot n})$ of $\operatorname{pos} C(\alpha)$ the sign $(-1)^{|\alpha|}$. This induces orientations for the boundary pseudomanifolds. See, for example, Freund (1980).

□

As a final remark, it should be mentioned that the boundary pseudomanifolds need not be distinct. For example, Figure 4.5 shows $\mathrm{K}(M)$ for

$$M = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

In this case $M \in \mathrm{INS}_4$, and the four pseudomanifolds are the four complementary cones. Each has a different boundary pseudomanifold. Figure 4.6 shows $\mathrm{K}(M)$ for

$$M = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Here $M \in \mathrm{INS}_2$, and the two pseudomanifolds are $\{\operatorname{pos} C(\emptyset), \operatorname{pos} C(\{2\})\}$ and $\{\operatorname{pos} C(\{1\}), \operatorname{pos} C(\overline{2})\}$. These both have the pseudomanifold $\{\operatorname{pos} I_{\cdot 2}$, $\operatorname{pos} -M_{\cdot 2}\}$ as boundary.

4.4 A Simple Class of INS-matrices

The relation between INS-matrices and other matrix classes will be discussed in Chapter 5, however, it seems appropriate at this point to introduce a simple subclass of INS. So saying, we have

DEFINITION 4.19 We say that a matrix A is in the class GNI (*G*eneralized *N*egative *I*dentity) if and only if

$$A \in \bigcup_{n \in \mathbf{Z}_+} \{ M \in \Re_{-}^{n \times n} : |\operatorname{supp} M_{\cdot i}| \leq 1, \text{ for all } i \in \overline{n} \},$$

i.e., each column of the matrix contains at most one non-zero entry, and, if it exists, this non-zero entry is negative.

Suppose $M \in$ GNI and $\alpha \in (\overline{n})$. If $\operatorname{pos} C(\alpha)$ is a full cone then $\operatorname{pos} C(\alpha) = \Re_{+}^{n}$. Otherwise, $\operatorname{pos} C(\alpha) \subseteq \partial \Re_{+}^{n}$. Thus no face of any complementary cone intersects the interior of $\mathrm{K}(M) = \Re_{+}^{n}$. Accordingly, $\operatorname{int} \mathrm{K}(M) = \operatorname{int} \Re_{+}^{n}$ is itself one of the connected components in Σ. So, by Theorem 3.8,

$$\mathit{GNI} \subseteq \mathit{INS}.$$

GNI-matrices satisfy the conclusion of the theorems of Section 4.2, and the proof sheds light on the combinatorial aspect of the subject. In fact, the theorem essentially follows from the next lemma which is an interesting combinatorial result by itself.

LEMMA 4.20 Suppose we are given n boxes, labelled $1, 2, \ldots, n$, and $2n$ balls, labelled $1, 2, \ldots, n, \tilde{1}, \tilde{2}, \ldots, \tilde{n}$. Suppose also that, for all $i \in \overline{n}$, ball i is in box i, whereas ball $\tilde{\imath}$ may be in any one, or none, of the boxes. Say that $(l_1, l_2, \ldots, l_n)$ is a *list* if for all $i \in \overline{n}$, l_i equals j or $\tilde{\jmath}$ for some $j \in \overline{n}$, and ball l_i is contained in box i. Say that a list is *proper* if for all $i \in \overline{n}$ there exists a $j \in \overline{n}$ such that $l_j \in \{ i, \tilde{\imath} \}$. Then the number of proper lists is a power of two.

Proof. This will be by induction on n. If $n = 1$, the number of proper lists is 2 or 1 depending on whether ball $\tilde{1}$ is, respectively, in or not in box 1.

The lemma is true in this case.

Assume the lemma is true for $1, \ldots, n-1$. We will show it true for n. Suppose some bail is in no box. We may assume it is ball i. Then any proper list $(l_1, l_2, \ldots, l_n)$ has $l_n = n$. Notice that $(l_1, l_2, \ldots, l_{n-1})$ is a proper list, and any such proper list can be extended to a complete proper list by adjoining $l_n = n$. Also the distribution of the balls $1, \ldots, n-1, \tilde{1}, \ldots, \widetilde{n-1}$ in the boxes $1, \ldots, n-1$ satisfies the conditions of the lemma. Thus by induction the number of proper lists $(l_1, \ldots, l_{n-1})$ is a power of two, and this equals the number of complete proper lists $(l_1, \ldots, l_n)$.

Assume all the balls $\tilde{1}, \ldots, \tilde{n}$ are each in some box. Suppose ball $\tilde{\imath}$ is in box i. Again, we may assume $i = n$. Then, as above, any proper list $(l_1, \ldots, l_{n-1})$ can be made into a complete proper list by adjoining either $l_n = n$ or by adding in $l_n = \tilde{n}$. (Notice that either is possible.) Also, any proper list $(l_1, \ldots, l_n)$ will have $l_n = n$ or $l_n = \tilde{n}$, hence $(l_1, \ldots, l_{n-1})$ is a proper list. As above, we may use induction to show that the number of proper lists $(l_1, \ldots, l_{n-1})$ is a power of two. Thus we have twice that number of complete proper lists. This is still a power of two.

Now suppose, for all $i \in \overline{n}$, that we have $\tilde{\imath}$ in some box, but not box i. Let $i_1, i_2, i_3, \ldots$ be a sequence defined by letting $i_1 = 1$ and saying that ball $\tilde{\imath}_j$ is in box i_{j+1} for all $j \in \mathbf{Z}_+$. Then the sequence must clearly repeat a number at some point, say $i_j = i_k$, such that $j < k$, and $i_j, i_{j+1}, \ldots, i_{k-1}$ are all distinct. We may assume that $j = 1$, that $3 \leq k \leq n+1$, and the sequence at the end of the last sentence is $1, 2, \ldots, k-1$. Let $(l_1, \ldots, l_n)$ be a proper list. If $l_1 = 1$, then as ball $\widetilde{k-1}$ is in box 1, we need to have $l_{k-1} = k-1$. As ball $\widetilde{k-2}$ is in box $k-1$, we need to have $l_{k-2} = k-2$. Continuing in this fashion we find that $(l_1, \ldots, l_{k-1})$ is $(1, \ldots, k-1)$. If

$l_1 \neq 1$, then we need $l_2 = \tilde{1}$. As $l_2 \neq 2$, we need that $l_3 = \tilde{2}$. Continuing in this fashion we have that $(l_1, \ldots, l_{k-1})$ is $(k \tilde{-} 1, \tilde{1}, \tilde{2}, \ldots, k \tilde{-} 2)$. Thus we must have that $(l_1, \ldots, l_n)$ equals $(1, \ldots, k-1)$ or $(k \tilde{-} 1, \tilde{1}, \ldots, k \tilde{-} 2)$. Notice that either one of these two will do, since having either one of these begin the complete proper list will force the rest of the proper list, $(l_k, \ldots, \tilde{n})$, to be selected from the set $\{k, \ldots, n, \tilde{k}, \ldots, \tilde{n}\}$. Hence given an "ending" to the proper list that works with either of the previous two "beginnings," the "ending" will work with both of the "beginnings." Furthermore, we see the "ending" is just a proper list for the boxes $k, \ldots, n$ using balls $k, \ldots, n, \tilde{k}, \ldots, \tilde{n}$, and any such proper list will do. By induction, the number of such proper lists for the "ending" is a power of two. Since there are two possible "beginnings," the number of complete proper lists is also a power of two. This completes the induction, and the lemma follows. $\square$

The above lemma translates almost immediately into the

THEOREM 4.21

$$\mathbf{GNI} \subseteq \bigcup_{p=0}^{\infty} \mathbf{INS}_{2^p}.$$

Proof. Let $M \in \mathrm{GNI} \cap \Re^{n \times n}$. With reference to Lemma 4.20, ball i corresponding to $I_{\cdot i}$ and ball $\tilde{\imath}$ corresponding to $-M_{\cdot i}$ for $i \in \bar{n}$. We say a ball is in box i if and only if the i^{th} component of the corresponding vector is nonzero. Thus there is a bijective correspondence between full complementary cones and proper lists, where the elements of a proper list correspond to the columns of a nondegenerate complementary matrix. Each of these full cones is equal to $\Re^n_+ = \mathrm{K}(M)$, and as Lemma 4.20 now tells us the number of

such cones is a power of two, we have $M \in \mathbf{INS}_{2^p}$ for some nonnegative integer p.

□

It seems bothersome to require that the nonzero entries in a GNI-matrix be negative. It would be preferable to work with the following class of matrices.

DEFINITION 4.22 We say that a matrix A is in the class **GI** if and only if

$$A \in \bigcup_{n \in \mathbb{Z}_+} \{ M \in \Re^{n \times n} : |\operatorname{supp} M_{\cdot i}| \leq 1, \text{ for all } i \in \overline{n} \},$$

i.e., each column of the matrix contains at most one non-zero entry.

Unfortunately, as seen at the end of Section 2.1, the matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

is not **INS**, but is in **GI**. See Figure 2.3. Thus **GI** $\not\subseteq$ **INS**. However, it is "close" enough to warrant investigation, and so we look at the following combinatorial lemma which is an extension of Lemma 4.20. (The proofs are almost identical so the proof of Lemma 4.23 will be given in less detail than necessary, but familiarity with the reasoning in the proof of Lemma 4.20 will be assumed.)

LEMMA 4.23 Suppose we are given $2n$ boxes which are labelled $1, 2, \ldots, n, 1', 2', \ldots, n'$, and $2n$ balls, labelled $1, 2, \ldots, n, \tilde{1}, \tilde{2}, \ldots, \tilde{n}$. Suppose also that, for all $i \in \overline{n}$, ball i is in box i, but ball $\tilde{i}$ can be in any one box, or no box at all. Say that $(b_1, b_2, \ldots, b_n)$ is a *box list* if, for each $i \in \overline{n}$, b_i is either i or i'. Furthermore, given a box list, we say that $(l_1, l_2, \ldots, l_n)$ is a *list* for the box list if, for each $i \in \overline{n}$, l_i equals j or $\tilde{j}$ for some $j \in \overline{n}$, and

l_i is contained in box b_i. Say that a list is *proper* if, for all $i \in \bar{n}$, there is a $j \in \bar{n}$ such that $l_j \in \{i, \tilde{i}\}$. Then, there exists a nonnegative integer p such that the number of proper lists associated with any box list is either zero or 2^p.

Proof. We will use induction on n. For $n = 1$ either $\tilde{1}$ is in no box (box 1), in which case box 1 has one (two) proper list(s) and box $1'$ has none, else $\tilde{1}$ is in box $1'$, in which case both boxes have one proper list. The lemma holds here.

Now assume the lemma holds for $1, \ldots, n-1$. We will show it true for n. Suppose some ball, say $\tilde{n}$, is in no box. Then any box list with at least one proper list must have $b_n = n$. Also, any proper list must have $l_n = n$. Similar to before, we find that the number of proper lists that are associated with box lists of the form $(b_1, \ldots, b_{n-1}, n)$ equals the number of proper lists $(l_1, \ldots, l_{n-1})$ for the associated box lists $(b_1, \ldots, b_{n-1})$ when we consider the embedded smaller problem for $n - 1$. The lemma then holds here by induction.

Assume all the balls $\tilde{1}, \tilde{2}, \ldots, \tilde{n}$ are in some box. If for some i, say $i = n$, we have ball $\tilde{n}$ in box n, then we will have a situation similar to the above. Any box list will have no proper lists if $b_n = n'$, and the number of proper lists of box lists in the form $(b_1, \ldots, b_{n-1}, n)$ equals *twice* the number of proper lists $(l_1, \ldots, l_{n-1})$ for the box lists $(b_1, \ldots, b_{n-1})$ when we consider the smaller embedded problem. (We would just add on $l_n = n$ and $l_n = \tilde{n}$ to get the two complete proper lists from the box lists of the smaller problem.) The lemma then holds here by induction.

If we had $\tilde{n}$ in box n' in the previous paragraph, then each box list $(b_1, \ldots, b_n)$ would have the same number of proper lists as $(b_1, \ldots, b_{n-1})$ in

the smaller embedded problem. The reasoning is the same as before, only now we complete the smaller proper lists by adding on $l_n = n$ if $b_n = n$, and $l_n = \tilde{n}$ if $b_n = n'$. The lemma will still hold by induction.

So now, finally, assume that all the balls $\tilde{1}, \ldots, \tilde{n}$ are in some box but, for each $i \in \overline{n}$, ball $\tilde{\imath}$ is neither in box i nor in box i'. Thus, as in the proof of Lemma 4.20, we may assume for some k, where $3 \leq k \leq n+1$, that, for all $i \in \overline{k-2}$, the ball $\tilde{\imath}$ is in either box $i+1$ or box $(i+1)'$, and the ball $\widetilde{k-1}$ is in either box 1 or box $1'$. Now suppose the box list $(b_1, \ldots, b_n)$ has a proper list $(l_1, \ldots, l_n)$. Suppose $l_1 = 1$. Thus $b_1 = 1$, so no other ball in the proper list can be from box 1 *or* box $1'$. Hence we need $l_{k-1} = k-1$. Continuing on in this fashion, as in the proof of Lemma 4.20, we get $(l_1, \ldots, l_{k-1})$ is $(1, \ldots, k-1)$. If $l_1 \neq 1$, then we need $l_2 = \tilde{1}$. Thus b_2 equals whichever of the two boxes, 2 or $2'$, contains $\tilde{1}$. In either case, we cannot select any other ball from either of the two boxes for the proper list, hence we need $l_3 = \tilde{2}$, and continuing on we have $(l_1, \ldots, l_{k-1})$ is $(\widetilde{k-1}, \tilde{1}, \tilde{2}, \ldots, \widetilde{k-2})$. Each of these cases determines the "beginning" of the box list, i.e., $(b_1, b_2, \ldots, b_{k-1})$. If the list in each case is the same, then any box list which has at least one associated proper list must have this "beginning." The ending, as in Lemma 4.20, can be any proper list $(l_k, \ldots, l_n)$ from the smaller embedded problem. By induction this is a fixed power of two, say 2^p, and so the number of complete proper lists, for any box list having proper lists, is 2^{p+1}. If the two "beginnings" are different, then any box list having proper lists must have one of these two "beginnings," and the number of proper lists it will have will be 2^p.

In all cases, all box lists with proper lists have the same number of proper lists, and that number is a power of two. The induction is now completed.

□

Now, with this lemma, we can finish this section with the following

THEOREM 4.24 If $M \in \mathrm{GI} \cap \Re^{n \times n}$, there exists a nonegative integer p, such that for all $q \in \mathrm{K}(M)$

$$|\operatorname{supp} q| = n \quad \Rightarrow \quad |\mathrm{sol}(q, M)| = 2^p .$$

Proof. With reference to Lemma 4.23, let ball i correspond to $I_{\cdot i}$, and ball $\bar{\imath}$ correspond to $-M_{\cdot i}$. We say a ball is in box i if and only if the corresponding vector has it's i^{th} component positive. We say a ball is in box i' if and only if the corresponding vector has it's i^{th} component negative. Each full complementary cone must be, geometrically, an orthant in $\Re^n$. Each degenerate complementary cone must be, geometrically, contained in the union of the boundaries of the orthants. There is a bijective correspondence between orthants and box lists. Thus the interior of each orthant that is contained in $\mathrm{K}(M)$ must be an element in Σ. Since there is a bijective correspondence between full cones covering an orthant and proper lists of the orthant's associated box list, Lemma 4.23 implies that each orthant contained in $\mathrm{K}(M)$, i.e., with some associated proper list, must be covered by the same number of full cones as the other orthants in $\mathrm{K}(M)$, and that number must be a power of two. Thus, by Theorem 3.8, $|\mathrm{sol}(q, M)|$ is this power of two for any q belonging to $\mathrm{K}(M)$ and the interior of an orthant.

□

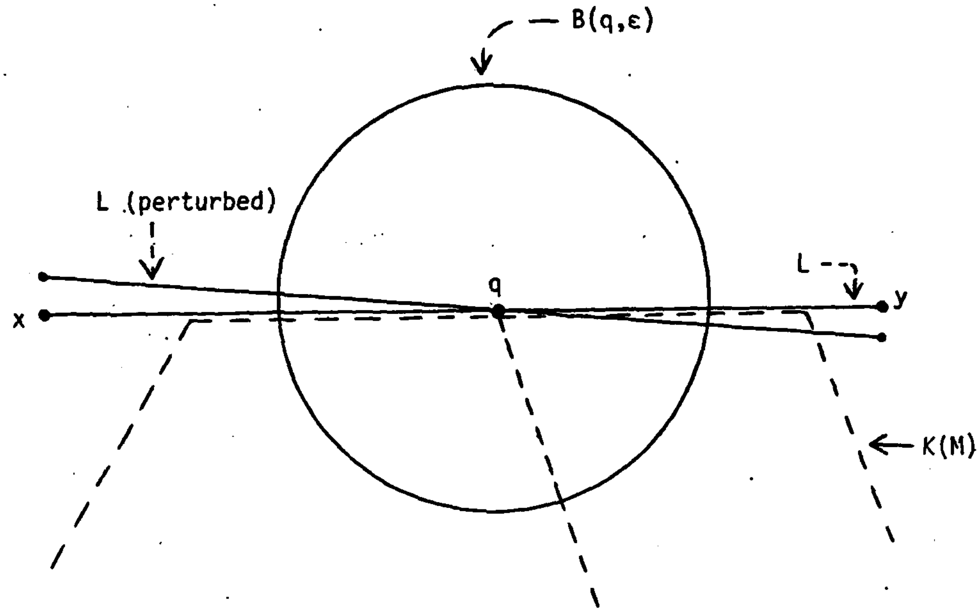


Figure 4.1

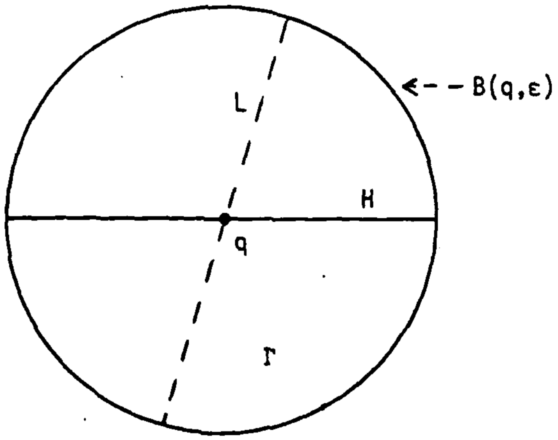


Figure 4.2

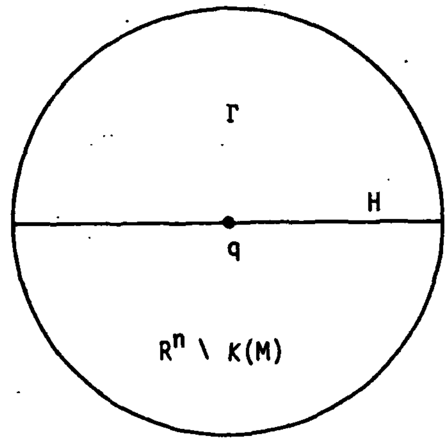


Figure 4.3

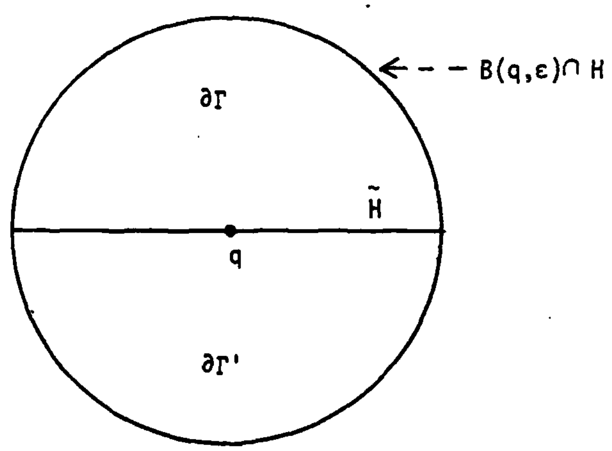


Figure 4.4

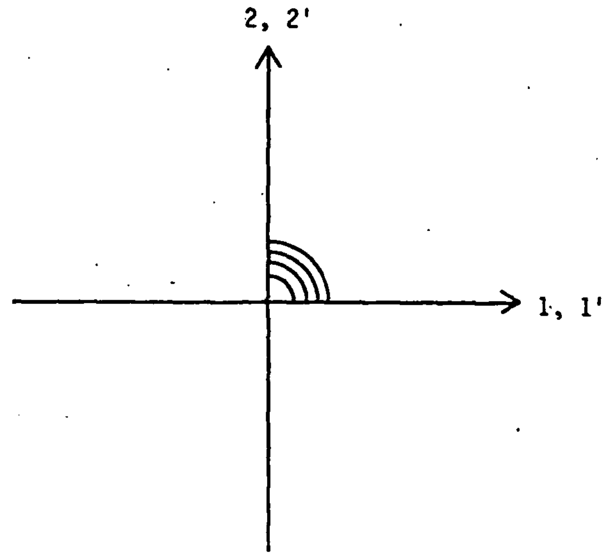


Figure 4.5

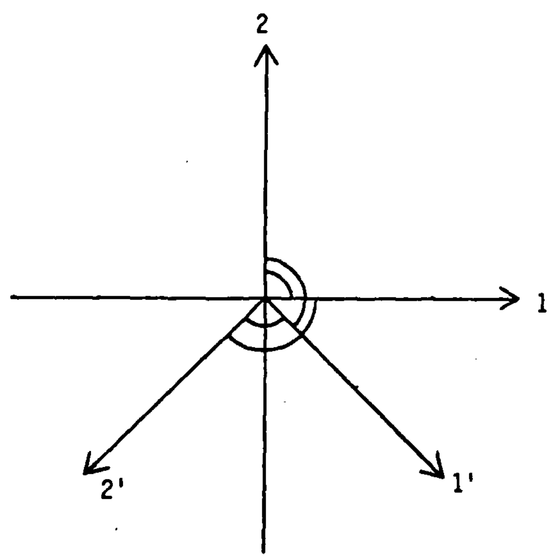


Figure 4.6

CHAPTER 5.
MATRIX CLASSES AND LCP THEORY

5.1 Matrix Classes

Much of the literature concerning the **LCP** deals with the study of matrix classes. Some classes are defined using the **LCP** itself and so we seek more constructive characterizations. Other classes are defined using more simple and testable criteria and results are found concerning the nature of the **LCP** (q, M) when M is in one of these classes. The relationships among the classes has also been a rich subject of study, and much work has been devoted to trying to understand which basic properties of importance to the **LCP** are common, or different, among the matrix classes. In Figure 5.1 we have listed the seven matrix classes defined in this work along with some of the more well-studied matrix classes in the field. This figure should be referred to throughout this section. (The arrows indicate inclusion relationships among the classes, with the larger classes tending to be at the top of the page.) The purpose of this section is to define the classes in this "family tree," and to discuss just where **U** and **INS** fit into it. There is no attempt to give a detailed review of these classes, however references are given showing where

more information can be obtained. Some basic references of general value are Lemke (1970), Karamardian (1972), Kostreva (1976), Mohan (1978), and Cottle (1983). The classes are presented in alphabetical order by the symbols used in Figure 5.1. At times it will be necessary to refer to the definition of a matrix class not yet given.

(A) A matrix $M \in \Re^{n \times n}$ is said to be *adequate*, $M \in \mathbf{A}$, if and only if $M \in \mathbf{P}_0$ and for all $\alpha \in (\overline{n})$ we have $\det M_{\alpha\alpha} = 0$ implies the column vectors $M_{\cdot\alpha}$ are linearly dependent and the row vectors $M_{\alpha\cdot}$ are linearly dependent. See Ingleton (1966), Cottle (1968) and Eaves (1971).

(BG) A matrix $M \in \Re^{n \times n}$ is said to be a *bimatrix game* matrix, $M \in \mathbf{BG}$, if and only if for some $m \in (\overline{n-1})$ there are matrices $A \in \Re^{m \times (n-m)}$ and $B \in \Re^{(n-m) \times m}$ where $A, B > 0$ and

$$M = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}.$$

See Lemke and Howson (1964), Cottle and Dantzig (1968) and Eaves (1971).

(CP) A matrix $M \in \Re^{n \times n}$ is said to be *copositive*, $M \in \mathbf{CP}$, if and only if for all $x \in \Re^n$, $x \geq 0$ implies $x^T M x \geq 0$. This matrix class has also been denoted as $\mathbf{C}_0$. Copositive matrices are important in combinatorics and other fields aside from complementarity. There is a large literature about this class, for example, see Gaddum (1958), Cottle and Dantzig (1968), Cottle, Habetler, and Lemke (1970b), Pereira (1972), Hoffman and Pereira (1973), and Evers (1978).

(C+) A matrix $M \in \Re^{n \times n}$ is said to be *copositive-plus*, $M \in \mathbf{C+}$, if and only if M is copositive and for all $x \in \Re^n$, $x \geq 0$ and $x^T M x = 0$ imply $(M + M^T)x = 0$. Like the copositive matrices, there is a large literature concerned with these matrices. See the papers given as references for the

copositive matrices.

($\mathbf{E}_0$) A matrix $M \in \Re^{n \times n}$ is said to be *semi-monotone*, $M \in \mathbf{E}_0$, if and only if for all $x \in \Re^n$, $0 \neq x \geq 0$ implies there is some $k \in \overline{n}$ such that $x_k > 0$ and $(Mx)_k \geq 0$. (This class has also been denoted as $\mathbf{L}_1$.) If M is symmetric, then M is semi-monotone if and only if M is copositive. We have used these matrices previously, with their other characterization of being the class of matrices M for which $|\mathrm{sol}(q, M)| = 1$ for all $0 < q \in \Re^n$. Like the copositive matrices, the semi-monotone matrices have been extensively studied. See, for example, Lemke (1970), Eaves (1971), Pereira (1972), Karamardian (1972), and Garcia (1973).

($\mathbf{E}_0^{\mathrm{f}}$) A matrix $M \in \Re^{n \times n}$ is said to be *fully semi-monotone* if and only if all principal transforms of M are semi-monotone. This matrix class was introduced in this work, and was shown to contain the matrix classes $\mathbf{U}$ and $\mathbf{P}_0$. (It is clearly contained in $\mathbf{E}_0$.) As seen, it can be characterized as the class of matrices such that for all $q \in \Re^n$, if $(w, z) \in \mathrm{sol}(q, M)$ and $w + z > 0$ then $\{(w, z)\} = \mathrm{sol}(q, M)$. To see why $\mathbf{E}_0^{\mathrm{f}}$ has been placed where it is in Figure 5.1, consider the matrices

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \quad \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \quad \begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(5.1) (5.2) (5.3) (5.4) (5.5)

None of these matrices are in $\mathbf{E}_0^{\mathrm{f}}$. However, (5.1) is in **SCP**, **E**, and $(\mathbf{N} \cap \mathbf{Q})^{-1}$, (5.2) is in $\overline{\mathbf{Z}}$, (5.3) is in $\mathbf{N} \cap \mathbf{Q}$, (5.4) is in $\mathbf{N} \setminus \mathbf{Q}$, and (5.5) is in **GNI**. Consider now

$$M = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \\ 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}. \tag{5.6}$$

M *is in* **BG**, *but is not in* $\mathbf{E}_0^f$. This can be seen as we have

$$M^{-1} = \begin{bmatrix} 0 & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{2}{3} & -\frac{1}{3} & 0 & 0 \end{bmatrix}$$

and the inverse of a matrix, if it exists, is always a principal transform. However, with $x = (1,0,1,0)^T$, we note that there is no index $k \in \bar{4}$ for which $x_k > 0$ and $(M^{-1}x)_k \geq 0$. Hence $M \notin \mathbf{E}_0^f$.

(E) A matrix $M \in \Re^{n \times n}$ is said to be *strictly semi-monotone* if and only if for all $x \in \Re^n$, $0 \neq x \geq 0$ implies there is some $k \in \bar{n}$ such that $x_k > 0$ and $(Mx)_k > 0$. (This class has also been denoted as $\mathbf{L}_*$.) If M is symmetric, then M is strictly semi-monotone if and only if M is strictly copositive. Similar to the semi-montone matrices, these matrices can be characterized as being the class of matrices M for which $|\mathrm{sol}(q, M)| = 1$ for all $0 \leq q \in \Re^n$. See the papers given as references for the semi-monotone matrices. This matrix class is also the class of *completely* **Q**-matrices, which are defined to be those **Q**-matrices all of whose principal submatrices are also **Q**-matrices. This equivalence was shown by Cottle (1979).

(GI) A matrix $M \in \Re^{n \times n}$ is said to be in **GI** if and only if for all $i \in \bar{n}$ we have $|\mathrm{supp}\, M_{\cdot i}| \leq 1$. This class was brought up in Chapter 4 due to its combinatorial nature, and because it is "almost" in the class **INS**. For such a simple class of matrices, it seems surprising that it is contained in *none* of the other matrix classes in Figure 5.1. Still, Example 2.3 is a **GI**-matrix that is not in $\mathbf{Q}_0$, and the 1×1 matrix (these are usually referred to as "numbers") $[-1]$ is not in $\mathbf{S}_0$. As mentioned in Chapter 4, $\mathbf{GI} \not\subseteq \mathbf{INS}$. The class **GI** is contained in no other matrix class in Figure 5.1, since every other

matrix class shown there is a subclass of $\mathbf{Q}_0$, $\mathbf{S}_0$, or **INS**.

(GNI) A matrix $M \in \Re^{n\times n}$ is said to be in GNI if and only if $M \in \mathbf{GI}$ and $M \leq 0$. It was shown in Chapter 4 that this class is in INS. In fact, $M \in \mathbf{INS}_k \cap \mathbf{GNI}$ implies $k = 2^p$ for some nonnegative integer p. Also $\mathbf{INS}_{2^p} \cap \mathbf{GNI} \neq \emptyset$ for all nonnegative integers p, as the zero matrix is in $\mathbf{INS}_1$ and $-I \in \Re^{p\times p}$ is in $\mathbf{INS}_{2^p}$.

(INS) A matrix $M \in \Re^{n\times n}$ is said to be in INS, for *Invariant Number of Solutions*, if and only if there is some positive integer k such that for all $q \in \operatorname{int} \mathrm{K}(M)$ we have $|\mathrm{sol}(q, M)| = k$. We have studied these matrices a great deal. Notice now where they fit into Figure 5.1. We know from Theorem 3.4 that $\mathbf{INS} \cap \mathbf{Q} = \mathbf{P}$. Also, it is shown in Garcia (1973) that $M \in \mathbf{L}(d)$ with $d > 0$ implies $|\mathrm{sol}(d, M)| = 1$, and hence we have

$$\bigcup_{d>0} \mathbf{L}(d) \cap \mathbf{INS} = \mathbf{U}.$$

We see that $\mathbf{E}_0 \cap \mathbf{INS} = \mathbf{U}$, since for $M \in \mathbf{E}_0$ we have $|\mathrm{sol}(q, M)| = 1$ for all $q > 0$. More will be said about INS matrices in relation to some of the other classes, but, before moving on, notice that the matrix given in (5.5) is in INS but is not in $\mathbf{S}_0$, so $\mathbf{INS} \not\subseteq \mathbf{S}_0$.

(K) A matrix $M \in \Re^{n\times n}$ is said to be in K if and only if $M \in \mathbf{P} \cap \mathbf{Z}$. (These matrices have also been referred to as the Minkowski matrices and denoted as the class M.) These matrices have a great deal of structure, both geometric and algebraic. It is interesting to note $\mathbf{K} = \mathbf{Z} \cap \mathbf{Q}$, i.e., the complementary cones of a Z-matrix cover $\Re^n$ if and only if they partition $\Re^n$. (The meaning of "partition" allows the cones to intersect on their boundaries.) The classic reference for these matrices is Fiedler and Pták (1962). See also Cottle and Veinott (1972).

($\mathbf{K_0}$) A matrix $M \in \Re^{n\times n}$ is said to be in $\mathbf{K_0}$ if and only if $M \in \mathbf{P_0} \cap \mathbf{Z}$. Again, the classic reference here is Fiedler and Pták (1962). In Mohan (1980), it is shown that the boundary of a $\mathbf{K_0}$ matrix is the union of the degenerate faces. Since, for $M \in \mathbf{K_0}$, there are no reflecting faces in $\mathrm{K}(M)$, as $\mathbf{K_0} \subseteq \mathbf{P_0}$, it follows that $\mathrm{K}(M)$ is regular. In Chandrasekaran (1970) it was shown that $\mathbf{Z} \in \mathbf{Q_0}$, hence $\mathrm{K}(M)$ is convex for a $\mathbf{K_0}$-matrix, and so $\operatorname{int}\mathrm{K}(M)$ will be connected. Thus $\mathbf{K_0} \subseteq \mathbf{INS}$ by Corollary 3.14. As $\mathbf{K_0} \subseteq \mathbf{P_0} \subseteq \mathbf{E_0}$, we see we must have $\mathbf{K_0} \subseteq \mathbf{U}$. In Mohan (1980), other results are derived about $\mathbf{K_0}$ which can be viewed as consequences of some of the theorems presented here concerning U-matrices. See also Mohan (1978) for more on $\mathbf{K_0}$-matrices.

($\mathbf{L}$) A matrix $M \in \Re^{n\times n}$ is said to be in $\mathbf{L}$ if and only if $M \in \mathbf{E_0}$, and for all $(w,z) \in \mathrm{sol}(0,M)$, where $z \neq 0$, there is a $x \in \Re^n$, $0 \neq x \geq 0$, with $z \geq x$ and $w \geq -M^T x \geq 0$. This is one of the largest classes of matrices that Lemke's algorithm using $e = (1,1,\ldots,1)$ is known to process. The standard reference for this class, which is also the reference defining the class, is Eaves (1971).

($\mathbf{L}(d)$) A matrix $M \in \Re^{n\times n}$ is said to be in $\mathbf{L}(d)$ if and only if for all $(w,z) \in \mathrm{sol}(\lambda d, M)$, where $z \neq 0$ and $\lambda \geq 0$, there is a $x \in \Re^n$, $0 \neq x \geq 0$, with $z \geq x$ and $w \geq -M^T x \geq 0$. The standard reference for these classes is Garcia (1973). It should be pointed out that $\mathbf{L} = \bigcap_{d>0} \mathbf{L}(d)$.

($\mathbf{L}^*(d)$) A matrix $M \in \Re^{n\times n}$ is said to be in $\mathbf{L}^*(d)$ if and only if for all $\lambda \geq 0$ we have that $(w,z) \in \mathrm{sol}(\lambda d, M)$ implies $(w,z) = (\lambda d, 0)$. For $d \geq 0$, $\mathbf{L}^*(d)$ is the class of all matrices M where in $\mathrm{K}(M)$ the only complementary cones containing d are $\operatorname{pos} C(\alpha)$ where $\alpha \cap \operatorname{supp} d = \emptyset$, and there are no strongly degenerate cones in $\mathrm{K}(M)$. For $d \not\geq 0$, $\mathbf{L}^*(d)$ is

the class of all matrices M where $\mathrm{K}(M)$ has no strongly degenerate cones and does *not* contain d. These classes, as well as the $\mathbf{L}(d)$, were introduced in Garcia (1973). There it is shown if $M \in \mathbf{L}(d)$, with $d > 0$, then for all $\lambda > 0$, we have $\{(\lambda d, 0)\} = \mathrm{sol}(\lambda d, M)$. Hence, for $d > 0$, we have $\mathbf{L}^*(d) = \mathbf{L}^*(0) \cap \mathbf{L}(d)$. While before we had $\mathbf{L} = \cap_{d>0} \mathbf{L}(d)$, we can only say here that $\mathbf{E} \subseteq \cap_{d>0} \mathbf{L}^*(d)$. For example, the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

is in $\cap_{d>0} \mathbf{L}^*(d)$, but is *not* in $\mathbf{E}$. We will have more to say about these classes later on.

(N) A matrix $M \in \Re^{n \times n}$ is said to be in N if and only if all principal minors are negative. Two standard references for this class are Saigal (1972a), and Kojima and Saigal (1979). More will be said about this class in what follows.

$(\mathbf{N} \cap \mathbf{Q})$ A matrix $M \in \Re^{n \times n}$ is in this class if and only if it is both in N and in $\mathbf{Q}$. It is shown in Kojima and Saigal (1979) that if $M \in \mathbf{N}$, then $M \in \mathbf{Q}$ if and only if $M \not< 0$. It is also shown if $M \in \mathbf{N} \cap \mathbf{Q}$, then $|\mathrm{sol}(q, M)|$ equals 1 for $q \not\geq 0$, and equals 2 for $0 \not< q \geq 0$. According to Theorem 3.3 of Kojima and Saigal (1979), if $M \in \mathbf{N} \cap \mathbf{Q}$ and $q > 0$ then $|\mathrm{sol}(q, M)|$ equals 3 if all solutions to (q, M) are nondegenerate, and equals 2 otherwise. (Actually, what it means for a solution of (q, M) to be "nondegenerate" is never defined in that paper, however, it can be inferred from context and the cited references that the intended definition is the one given here in Chapter 1.) While it is true that there will be exactly three solutions for all $q > 0$ having only nondegenerate solutions, it is true that there are exactly three solutions for *all* $q > 0$. The last line of the

proof, given in Kojima and Saigal (1979), says a solution is "lost" because of degeneracy. This will be the case when $0 \not< q \geq 0$ and it is on a reflecting face; however, for $q > 0$ we are only contending with proper faces, and no solutions are "lost." Consider

EXAMPLE 5.1 Let

$$M = \begin{bmatrix} -1 & 4 & 1 \\ 1 & -1 & -4 \\ 2 & -1 & -1 \end{bmatrix}.$$

It can be easily checked that $M \in \mathbf{N}$ and, as $M \not< 0$, $M \in \mathbf{Q}$. Let $q = (2, 4, 1)^T$. Then (q, M) has three solutions

$$\begin{aligned} (w^1, z^1) &= (2, 4, 1, 0, 0, 0) \\ (w^2, z^2) &= (0, 6, 5, 2, 0, 0) \\ (w^3, z^3) &= (3, 0, 0, 0, 0, 1) \end{aligned}$$

and (w^3, z^3) is degenerate.

$(\mathbf{N} \setminus \mathbf{Q})$ A matrix $M \in \Re^{n\times n}$ is in this class if and only if M is in $\mathbf{N}$ but *not* in $\mathbf{Q}$. In Kojima and Saigal (1979) it is shown that this is the set of matrices $M \in \mathbf{N}$ for which $M < 0$, hence, as pointed out in the paper, we will have $\mathrm{K}(M) = \Re^n_+$. Therefore these matrices are in $\mathbf{Q_0}$, hence all of $\mathbf{N}$ is in $\mathbf{Q_0}$. Kojima and Saigal (1979) also shows, for all $q > 0$, i.e, all $q \in \operatorname{int} \mathrm{K}(M)$, we have $|\mathrm{sol}(q, M)| = 2$. This means

$$\mathbf{N} \setminus \mathbf{Q} \subseteq \mathbf{INS_2}.$$

$((\mathbf{N} \cap \mathbf{Q})^{-1})$ A matrix $M \in \Re^{n\times n}$ is in this class if its inverse is in $\mathbf{N} \cap \mathbf{Q}$. This is equivalent to saying the matrix is in $\mathbf{Q}$, its determinant

is negative, and all of its proper principal submatrices are in **P**. Thus all proper principal submatrices are in **Q**, along with the matrix itself. Hence these matrices are completely-**Q**, which is to say they are in **E**. For more on these matrices see Saigal (1972b). The following example helps to justify the placement of these matrices in Figure 5.1. Let

$$M = \begin{bmatrix} 3 & -8 & 0 \\ -1 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}.$$

Notice $M^{-1} \in \mathbf{N} \cap \mathbf{Q}$. However, $M \notin \mathbf{CP}$ as letting $x = (3,2,0)^T$ we have $x^T M x < 0$.

(P) A matrix $M \in \Re^{n \times n}$ is in the class **P** if and only if all principal minors of M are positive. This is one of the most studied classes of matrices related to the LCP. There are many equivalent characterizations of these matrices, for example: $M \in \mathbf{P}$ if and only if for all $q \in \Re^n$ we have $|\mathrm{sol}(q, M)| = 1$, see Samelson, Thrall and Wesler (1958), and Murty (1972); $M \in \mathbf{P}$ if and only if, for $x \in \Re^n$, we have $x_i(Mx)_i \leq 0$ for all $i \in \bar{n}$ implies $x = 0$, see Fiedler and Pták (1962), also Gale and Nikaido (1965); and $M \in \mathbf{P}$ if and only if, for $\Lambda \in \Re^{n \times n}$, we have $\det(I - \Lambda + \Lambda M) \neq 0$ for all $0 \leq \Lambda \leq I$, see Aganagic (1978). The middle characterization gives some intuition behind the definition of **E**, as it states a matrix belongs to **P** if and only if for every non-zero $x \in \Re^n$, (not just $x \in \Re^n_+$), we have an index $k \in \bar{n}$ for which $x_k(Mx)_k > 0$. An interesting characterization by Habetler and Kostreva (1980) is as follows. Say a point $x \in \Re^n$ is a *complementary* point of (q, M) if and only if there is a $z \in \Re^n$, where for all $i \in \bar{n}$ we have $(Mz + q)_i z_i = 0$, such that $x = z + (Mz + q)$. It is then the case that $M \in \mathbf{P}$ if and only if there is some $q \in \Re^n$ such that the interior of each orthant in $\Re^n$ contains exactly one complementary point of (q, M).

For more on **P**-matrices, see the references mentioned and also Fiedler and Pták (1966).

($\mathbf{P}_0$) A matrix $M \in \Re^{n\times n}$ is in $\mathbf{P}_0$ if and only if all principal minors of M are nonnegative. Like **P**, this class has been extensively studied. In fact, the question of exactly what structure and properties are lost when dealing with $\mathbf{P}_0$ as opposed to dealing with **P** was one of the questions leading to the present work, and to other works in the field. Again, major references to this class are the papers by Fiedler and Pták (1962, 1966). An interesting characterization of $\mathbf{P}_0$, giving insight into the definition of $\mathbf{E}_0$, comes from Fiedler and Pták (1966) and states $M \in \mathbf{P}_0$ if and only if for all $0 \neq x \in \Re^n$, (not just $x \in \Re^n_+$), we have an index $k \in \bar{n}$ for which $x_k \neq 0$ and $x_k(Mx)_k \geq 0$. We move on to a special class of $\mathbf{P}_0$-matrices which were defined earlier in this work.

($\mathbf{P}_1$) A matrix $M \in \Re^{n\times n}$ is in $\mathbf{P}_1$ if and only if $M \in \mathbf{P}_0$ and exactly one principal minor of M is zero. This class fits into Figure 5.1 in about the same position as $\mathbf{P}_0$. However, we do know

THEOREM 5.2 $\mathbf{P}_1 \subseteq \mathbf{L}$.

Before starting the proof, we introduce a lemma.

LEMMA 5.3 If $M \in \mathbf{E}_0 \cap \Re^{n\times n}$ and for some $i \in \bar{n}$ we have $M_{\cdot i} \geq 0$ with $M_{ii} = 0$, then $M \notin \mathbf{Q}$.

Proof. Suppose we have a matrix M satisfying the hypothesis of the lemma. Take some $q \in \operatorname{pos} C(\emptyset)_{\cdot\bar{i}}$. By reasoning similar to previous dimensional arguments, we may assume q lies in no k-dimensional faces of $\mathrm{K}(M)$, for $k < n-1$, and any $(n-1)$-dimensional face of $\mathrm{K}(M)$ that contains q is contained in the hyperplane

$$H = \operatorname{span} C(\emptyset)_{\cdot i} = \{\, x \in \Re^n : x_i = 0 \,\}.$$

Let H^+ be the closed half-space with H as boundary that contains $I_{\cdot i}$, and let H^- be the other closed half-space. Let $\operatorname{pos} C(\alpha)_{\cdot \bar{j}}$ be a $(n-1)$-dimensional face of $\mathrm{K}(M)$ that contains $M_{\cdot i}$ and is contained in H. We see $\operatorname{pos} C(\alpha)_{\cdot \bar{j}}$ cannot contain $I_{\cdot i} \notin H$. Also, the vectors of $C(\alpha)_{\cdot \bar{j}}$ are linearly independent, as $\operatorname{pos} C(\alpha)_{\cdot \bar{j}}$ is $(n-1)$-dimensional, hence the face cannot contain $-M_{\cdot i}$. Thus $j = i$. As $M_{\cdot i} \in \operatorname{pos} C(\emptyset)_{\cdot \bar{i}}$, and as the $(n-1)$-dimensional faces of $\mathrm{K}(M)$ are finite in number and closed, we can select q close enough to $M_{\cdot i}$ such that we have the additional property that any $(n-1)$-dimensional face, $\operatorname{pos} C(\alpha)_{\cdot \bar{j}}$, of $\mathrm{K}(M)$ that contains q must have $j = i$. Now for all $\epsilon > 0$ small enough, $B(q,\epsilon) \cap K(M) = B(q,\epsilon) \cap H$, and no face of $\mathrm{K}(M)$ whose dimension is smaller than $n-1$ intersects $B(q,\epsilon)$. Thus $B(q,\epsilon) \cap \operatorname{pos} C(\emptyset)_{\cdot \bar{i}} = B(q,\epsilon) \cap H$. Hence

$$\emptyset \neq B(q,\epsilon) \cap \operatorname{int} \operatorname{pos} C(\emptyset) \subseteq H^+.$$

Since $M \in \mathbf{E}_0$, no other full cone can intersect the interior of $\operatorname{pos} C(\emptyset)$. Thus any full cone, $\operatorname{pos} C(\alpha)$, containing $B(q,\epsilon) \cap \operatorname{int} H^-$ must have a boundary face in H. This face will then contain q, and so this face is $\operatorname{pos} C(\alpha)_{\cdot \bar{i}}$. As both $I_{\cdot i}$ and $-M_{\cdot i}$ are in H^+, then we have $\operatorname{pos} C(\alpha) \subseteq H^+$, giving us a contradiction. Hence no full cone, and hence no cone, contains $B(q,\epsilon) \cap H^- \neq \emptyset$. Thus $M \notin \mathbf{Q}$.

□

Proof of Theorem 5.2. Let $M \in \mathbf{P}_1 \cap \Re^{n \times n}$. We know $M \in \mathbf{E}_0$ as $\mathbf{P}_1 \subseteq \mathbf{P}_0 \subseteq \mathbf{E}_0^f \subseteq \mathbf{E}_0$. If $\operatorname{sol}(0, M) = \{(0,0)\}$ then $M \in \mathbf{L}$. Thus assume there is a non-trivial solution, say (w, z) with $z \neq 0$, to $(0, M)$. Thus, letting $y = w + z$, we have for some $\alpha \in (\overline{n})$ that $y_\alpha = z_\alpha$, $y_{\bar{\alpha}} = w_{\bar{\alpha}}$, and

$C(\alpha)y = 0$. Thus we know $M_{\alpha\alpha} y_\alpha = 0$. In addition, we must have $y_\alpha > 0$ for otherwise some principal submatrix of $M_{\alpha\alpha}$, and hence of M, is singular, but $M_{\alpha\alpha}$ is the only singular principal submatrix of M. In the same way, we know $y_{\hat\alpha} > 0$. Else, for some $i \in \hat\alpha$, we have $M_{\{i\}\alpha} y_\alpha = 0$. Thus, with $\beta = \alpha \cup \{i\}$, we have $M_{\beta\beta} z_\beta = 0$ which, again, contradicts the fact that $M_{\alpha\alpha}$ is the only singular principal submatrix. Thus $y > 0$.

We now show $M \notin \mathbf{Q}$. If $|\alpha| = 1$, then it must be that, for some $i \in \overline{n}$ where $\alpha = \{i\}$, we have $M_{\cdot i} \geq 0$ and $M_{ii} = 0$. Thus, by Lemma 5.3, $M \notin \mathbf{Q}$. Suppose $|\alpha| > 1$. Pick some $\beta \subseteq \alpha$ with $|\beta| = |\alpha| - 1$, and let $\overline{M}$ be the principal transform of M gotten by block pivoting on $M_{\beta\beta}$. (Again, we know $M_{\beta\beta}$ is nonsingular as $M_{\alpha\alpha}$ is the only singular principal submatrix.) Since we have $M_{\alpha\alpha} y_\alpha = 0$ and $M_{\hat\alpha\alpha} y_\alpha > 0$, then, letting $\{i\} = \alpha \setminus \beta$, we have $\overline{M}_{\hat{i}i} > 0$ and $\overline{M}_{ii} = 0$. Since $M \in \mathbf{E}_0^{\mathbf{f}}$, we have $\overline{M} \in \mathbf{E}_0$, thus Lemma 5.3 gives us $\overline{M} \notin \mathbf{Q}$. Hence, as claimed, $M \notin \mathbf{Q}$.

From Theorem 2.25, $M \in \mathbf{U}$ and $\mathrm{K}(M)$ is a half-space. Let $0 \neq x \in \Re^n$ be a normal to the hyperplane $\partial\mathrm{K}(M)$. As $M \in \mathbf{U}$, so $\mathrm{K}(M)$ is regular, we must have $\operatorname{pos} C(\alpha) \subseteq \partial\mathrm{K}(M)$ thus $C(\alpha)^T x = 0$. Since all other complementary cones are full, they cannot be contained in $\partial\mathrm{K}(M)$. Thus $C(\hat\alpha)^T x > 0$. Therefore $x_\alpha > 0$ and $x_{\hat\alpha} = 0$. Also, $x^T M_{\cdot\alpha} = 0$ and $x^T M_{\cdot\hat\alpha} < 0$. Hence, we can choose x so that $\|x\|$ is so small that $z \geq x \geq 0$, and $w \geq -M^T x \geq 0$. This means M satisfies the conditions to be in $\mathbf{L}$, and the theorem follows.

□

It should be noted that $\mathbf{P}_1 \not\subseteq \mathbf{CP}$, for consider the matrix

$$M = \begin{bmatrix} 0 & -4 \\ 1 & 2 \end{bmatrix}.$$

Clearly, $M \in \mathbf{P}_1$. Yet, with $x = (1,1)^T$, we've $x^T M x < 0$.

($\mathbf{P}_1 \setminus \mathbf{Q}$) A matrix $M \in \Re^{n\times n}$ is in this "class" if and only if it is in $\mathbf{P}_1$, but not in $\mathbf{Q}$. This class has the same position in Figure 5.1 as does $\mathbf{P}_1$, except it is also contained in $\mathbf{U}$. More was said about $\mathbf{P}_1 \setminus \mathbf{Q}$ at the end of Chapter 2.

(PD) A matrix $M \in \Re^{n\times n}$ is said to be *positive definite*, $M \in \mathbf{PD}$, if and only if for all $0 \neq x \in \Re^n$ we have $x^T M x > 0$. For symmetric matrices, being in **P** is equivalent to being in **PD**, which is equivalent to there being some $L \in \Re^{n\times n}$ such that L is nonsingular and $M = L^T L$. For more concerning positive definite matrices, see Gantmacher (1960), Dantzig and Cottle (1967), Cottle, Habetler and Lemke (1970a), and Cottle (1983).

(PSD) A matrix $M \in \Re^{n\times n}$ is said to be *positive semi-definite*, $M \in \mathbf{PSD}$, if and only if for all $x \in \Re^n$ we have $x^T M x \geq 0$. For symmetric matrices, being in $\mathbf{P}_0$ is equivalent to being in **PSD**, which is equivalent to there being some $L \in \Re^{n\times n}$ such that $M = L^T L$. The class **PSD** is usually thought of in connection with convexity as the quadratic function $F(x) : \Re^n \to \Re$ defined by $F(x) = x^T M x + c^T x + d$, with $M \in \Re^{n\times n}$, $c \in \Re^n$ and $d \in \Re$, is convex if and only if $M \in \mathbf{PSD}$. See the references given for positive definite matrices.

(Q) A matrix $M \in \Re^{n\times n}$ is said to be in **Q** if and only if for all $q \in \Re^n$ the LCP (q, M) has at least one solution. This is equivalent to saying $\mathrm{K}(M) = \Re^n$. One of the major, and perhaps most difficult, problems in linear complementarity theory is to find a "good" characterization of **Q**, i.e., a characterization with which one could quickly test a matrix to determine whether or not it is in **Q**. Many of these other matrix classes were studied in attempts to find more classes of matrices that were contained in **Q**, or $\mathbf{Q}_0$.

Two interesting works concerning **Q** are Watson (1974), Kelly and Watson (1979). The latter contains a counterexample to a result of the former. In essence, it shows the annoying result that the set of **Q**-matrices is neither open *nor* closed in $\Re^{n \times n}$ for $n \geq 4$. Hence, the class **Q** will be hard to characterize. See also Cottle, von Randow and Stone (1981).

($\mathbf{Q_0}$) A matrix $M \in \Re^{n \times n}$ is said to be in $\mathbf{Q_0}$ if and only if for all $q \in \Re^n$ where the system of inequalities

$$Mz + q \geq 0 \qquad z \geq 0$$

is feasible, there exists at least one solution to the LCP (q, M). This is equivalent to saying $\mathrm{K}(M)$ is convex. Like **Q**, characterizing $\mathbf{Q_0}$ in a "good" way is a long standing problem. In fact, with a characterization of this class we can just say $\mathbf{Q} = \mathbf{Q_0} \cap \mathrm{S}$. Again, many of the works mentioned are concerned with $\mathbf{Q_0}$. For a recent and interesting paper on this class see Doverspike and Lemke (1981). (In other works, this class is denoted **K**; it should not be confused with the K used here.)

(**R**) A matrix $M \in \Re^{n \times n}$ is said to be *regular*, $M \in \mathbf{R}$, if and only if, with $e = (1, 1, \ldots, 1)^T \in \Re^n$, we have, for all $\lambda \geq 0$, that $\{(\lambda e, 0)\} = \mathrm{sol}(\lambda e, M)$. Clearly, $\mathbf{R} = \mathbf{L}^*(e)$. The standard reference for this class is Karamardian (1972). It is of interest to note, as shown by Aganagic and Cottle (1978), that $\mathbf{P_0} \cap \mathbf{Q} = \mathbf{P_0} \cap \mathbf{R}$. We cannot do better than this in classifying $\mathbf{P_0} \cap \mathbf{Q}$ as far as Figure 5.1 is concerned. For example, the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

is in **Q** and in $\mathbf{P_0}$ but is not in **E** which is the next matrix class "lower" in Figure 5.1 than **R**.

(S) A matrix $M \in \Re^{n \times n}$ is said to be in S if and only if there exists an $x \in \Re^n$ such that $x \geq 0$ and $Mx > 0$. This is the class of matrices for which (q, M) is "feasible" for all $q \in \Re^n$, i.e., for all $q \in \Re^n$ there is an $x_q \in \Re^n$ with $x_q \geq 0$ and $Mx_q + q \geq 0$, see Lemke (1970). The classic reference for these matrices is Fiedler and Pták (1966). Other relevant works to look at, that use S-matrices are Saigal (1971a) and Cottle (1979).

(S_0) A matrix $M \in \Re^{n \times n}$ is said to be in S_0 if and only if there exists an $x \in \Re^n$ such that $0 \neq x \geq 0$ and $Mx \geq 0$. This is clearly one of the largest matrix classes listed here, containing many of the others. Again, Fiedler and Pták (1966), Lemke (1970), and Saigal (1971a) are good references for this class. For a nice reference which extends the properties embodied in the matrix classes **P**, $\mathbf{P}_0$, S, and S_0 to non-linear functions, see Moré and Rheinboldt (1973).

Two of the inclusion arrows leading to the class S_0 in Figure 5.1 are not trivial, and have not been found by the author in the literature. The justification for these inclusions is in the following two theorems.

THEOREM 5.4 $\cup_{d>0} \mathbf{L}(d) \subseteq \mathrm{S}_0$.

Proof. Suppose for some $0 < d \in \Re^n$ we have $M \in \mathbf{L}(d) \cap \Re^{n \times n}$, but $M \notin \mathrm{S}_0$. If $(w, z) \in \mathrm{sol}(0, M)$, then $Mz \geq 0$ and $z \geq 0$. Thus $M \notin \mathrm{S}_0$ implies $z = 0$. Hence $\{(0,0)\} = \mathrm{sol}(0, M)$. Garcia (1973) shows that $M \in \mathbf{L}(d)$ implies for all $\lambda > 0$ that we have $\{(\lambda d, 0)\} = \mathrm{sol}(\lambda d, M)$. We conclude $M \in \mathbf{L}^*(d)$. But $\mathbf{L}^*(d) \subseteq \mathbf{Q} \subseteq \mathrm{S} \subseteq \mathrm{S}_0$, which gives us a contradiction. Thus $M \in \mathrm{S}_0$ and the theorem holds. □

THEOREM 5.5 $\mathbf{E}_0 \subseteq \mathrm{S}_0$.

Proof. Suppose $M \in \mathbf{E}_0 \cap \Re^{n\times n}$. Let $I \in \Re^{n\times n}$ be the identity matrix. Thus, for all $\epsilon > 0$, we have almost directly from the definitions that $M + \epsilon I \in \mathbf{E}$. Now $\mathbf{E} \subseteq \mathbf{Q} \subseteq \mathbf{S} \subseteq \mathbf{S}_0$, so for each $\epsilon > 0$ there is a $0 \neq x_\epsilon \geq 0$ such that $(M + \epsilon I)x_\epsilon \geq 0$. We may assume, by scaling, that $\|x_\epsilon\| = 1$. As the set

$$\{x \in \Re^n : \|x\| = 1\}, \qquad \text{the unit sphere in } \Re^n,$$

is compact, we have some point $x_0 \in \Re^n$, with $\|x_0\| = 1$, that is a cluster point of the set of x_ϵ. Thus, letting $\epsilon \to 0$, we see that $x_0 \geq 0$, and that $Mx_0 \geq 0$. Thus $M \in \mathbf{S}_0$, and the theorem holds. □

(SCP) A matrix $M \in \Re^{n\times n}$ is said to be *strictly copositive*, $M \in \mathbf{SCP}$, if and only if $x^T M x > 0$ for all $x \in \Re^n$ such that $0 \neq x \geq 0$. This class has also been denoted as **C**. For symmetric matrices, being in **SCP** is equivalent to being in **E**. See the references given for copositive matrices.

(U) A matrix $M \in \Re^{n\times n}$ is said to be in **U**, for *Unique solution*, if and only if for all $q \in \operatorname{int} \mathrm{K}(M)$ we have $|\mathrm{sol}(q, M)| = 1$. This matrix class was the topic of Chapter 2. If $M \in \mathbf{E}_0$, then for all $q > 0$ we have $|\mathrm{sol}(q, M)| = 1$. Garcia (1973) shows that if $M \in \mathbf{L}(d)$, for some $d > 0$, then $|\mathrm{sol}(d, M)| = 1$. Hence we see,

$$\mathbf{E}_0 \cap \mathbf{INS} = \mathbf{U} \qquad \left\{\bigcup_{d>0} \mathbf{L}(d)\right\} \cap \mathbf{INS} \subseteq \mathbf{U}. \tag{5.7}$$

Thus, as $\mathbf{U} \cap \mathbf{Q} = \mathbf{P}$, we have

$$\left\{\bigcup_{d>0} \mathbf{L}^*(d)\right\} \cap \mathbf{INS} = \mathbf{P}. \tag{5.8}$$

This helps us understand how $\mathbf{U}$ fits into Figure 5.1. Now consider the following matrices

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad\qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

(5.9) (5.10)

Notice (5.9) is contained in $\mathbf{U}$, but not in $\mathbf{Q}_0$. (This is Example 2.3.) Hence, the right side of (5.7) is a proper inclusion. As for (5.10), it is *not* in $\mathbf{U}$, yet it is in $\mathbf{A} \cap \mathbf{PSD} \cap \mathbf{SCP} \cap \mathbf{E} \cap \mathbf{P}_1$. Also, (5.6) showed an example of a matrix that was in $\mathbf{BG}$ but not in $\mathbf{E}_0^f$, hence certainly not in $\mathbf{U}$.

($\mathbf{Z}$) A matrix $M \in \Re^{n \times n}$ is said to be in $\mathbf{Z}$ if and only if for all $i, j \in \overline{n}$, where $i \neq j$, we have $M_{ij} \leq 0$. These matrices have been well studied. See, for example, Saigal (1971b) and the references mentioned in the paragraphs concerning the classes $\mathbf{K}$ and $\mathbf{K}_0$. In particular, see Mohan (1978).

($\overline{\mathbf{Z}}$) A matrix $M \in \Re^{n \times n}$ is said to be in $\overline{\mathbf{Z}}$ if and only if $M \in \mathbf{Z}$ and, for all $i \in \overline{n}$, we have $M_{ii} \geq 0$. (This class has also been denoted by $\mathbf{L}$.) See the references for $\mathbf{Z}$-matrices and, in particular, see Saigal (1972b) and Mohan (1978). One thing that should be pointed out is an error in Theorem 5.4 of Saigal (1972b). The theorem states that $M \in \overline{\mathbf{Z}}$ implies $\mathrm{K}(M)$ is regular. (Saigal's definition of regularity and the definition used in this work are different, however, all that need be known here is that the two definitions coincide for nondegenerate matrices.) This is *not* the case. Consider

$$M = \begin{bmatrix} 1 & -3 & -2 & 0 \\ -8 & 1 & 0 & -2 \\ -3 & 0 & 1 & -3 \\ -1 & -2 & -3 & 1 \end{bmatrix}.$$

This matrix is nondegenerate and contained in $\overline{\mathbf{Z}}$. Let $q = (51, 11, 19, 39)^T$ and $\tilde{q} = (49, 9, 21, 41)^T$. Then (q, M) and $(\tilde{q}, M)$ both only have non-degenerate solutions, but $|\mathrm{sol}(q, M)| = 4$ and $|\mathrm{sol}(\tilde{q}, M)| = 2$. Specifically, the solutions, (w, z), of (q, M) are

$$\begin{array}{cc} (51, 11, 19, 39, 0, 0, 0, 0) & (4, 0, 0, 0, 0, 8\tfrac{2}{3}, 10\tfrac{1}{2}, 9\tfrac{5}{6}) \\ (0, 0, 6\tfrac{4}{11}, 0, 3\tfrac{5}{11}, 18\tfrac{5}{33}, 0, \tfrac{25}{33}) & (0, 0, 0, 0, 1\tfrac{1}{3}, 13\tfrac{5}{9}, 5\tfrac{5}{6}, 6\tfrac{17}{18}) , \end{array}$$

while the solutions of $(\tilde{q}, M)$ are

$$(49, 9, 21, 41, 0, 0, 0, 0) \qquad (0, 0, 11\tfrac{2}{23}, 2\tfrac{19}{23}, 3\tfrac{7}{23}, 17\tfrac{10}{23}, 0, 0) .$$

This implies $M \notin \mathbf{INS}$ and, as M is nondegenerate, that $\mathrm{K}(M)$ is *not* regular by either definition. This incorrect result is referred to by Mohan (1978) in several places. Saigal (1972b) uses it to "show" that if $M \in \overline{\mathbf{Z}}$ then $M \in \mathbf{INS}_2$, which is clearly not true as seen in the example just given. More will be said about this in the next section.

5.2 Related LCP Theory

In this section we will consider some results in the LCP literature that seem related to the material we have covered. Most of the results concerning the exact number of solutions to the LCP have already been mentioned.

There is the classic result of Samelson, Thrall and Wesler (1958) that $\mathbf{P}$ is the set of all matrices M such that for all q we have $|\mathrm{sol}(q, M)| = 1$.

In Eaves (1971), it is shown if $M \in \mathbf{P}_0$ and q is contained in the interior of some full complementary cone then $|\mathrm{sol}(q, M)| = 1$. This can

be seen to follow from the fact that $\mathbf{P}_0 \subseteq \mathbf{E}_0^f$. For it is easily shown that $\mathbf{E}_0^f$ can be characterized as the set of those matrices M such that, for all q, if q is contained in the interior of a full complementary cone then $|\text{sol}(q, M)| = 1$. Related to this is Theorem 2.2 in Saigal (1970a) which states that $M \in \mathbf{P}_0$ implies that if $\text{sol}(q, M)$ contains a nondegenerate solution then $|\text{sol}(q, M)| = 1$. This can be seen to be in error by considering the matrix $[0] = M \in \mathbf{P}_0 \cap \Re^{1\times 1}$. We have for $\lambda > 0$ that $(w, z) = (0, \lambda)$ is a nondegenerate solution to $(0, M)$. It should also be mentioned that this result generalized a previous result of Lemke (1965) which used positive semi-definite matrices, a smaller subclass of $\mathbf{P}_0$.

There are several theorems by Murty (1972) on this subject, including another proof of the Samelson, Thrall and Wesler result. The main theorems from Murty (1972) of interest here are: $|\text{sol}(q, M)| < \infty$ for all q if and only if M is nondegenerate; if $|\text{sol}(q, M)|$ is constant over all non-zero q, then that constant is one and $M \in \mathbf{P}$; if $|\text{sol}(q, M)|$ is constant for all q which are nondegenerate with respect to M, then that constant is one.

The class N is studied in Kojima and Saigal (1979). It is shown that for $M \in \mathrm{N}$, if $M \not< 0$ then the value of $|\text{sol}(q, M)|$ is one for $q \not\geq 0$, is two for $0 \not< q \geq 0$, and is three for $q > 0$ nondegenerate with respect to M. This last part, as noted earlier, should state $|\text{sol}(q, M)| = 3$ for all $q > 0$. (Kojima and Saigal (1979) incorrectly state the value is two for $q > 0$ but degenerate with respect to M.) It is also shown for $M \in \mathrm{N}$, if $M < 0$ then the value of $|\text{sol}(q, M)|$ is zero for $q \not\geq 0$, is one for $0 \not< q \geq 0$, and is two for $q > 0$.

In Mohan (1980), it is shown for $M \in \mathbf{K}_0$, $q \in \text{int}\,\mathrm{K}(M)$ implies $|\text{sol}(q, M)| = 1$ and $q \in \partial\mathrm{K}(M)$ implies $|\text{sol}(q, M)| = \infty$.

In Saigal (1972b), the concept of a "regular pseudomanifold" was discussed with reference to $\mathrm{K}(M)$. $\mathrm{K}(M)$ was defined there as being "regular" if and only if every face was either proper or contained in $\partial \mathrm{K}(M)$. However, the definition of "proper" given there is different from what is used here. A face was defined there as being proper if and only if it is the intersection of the two adjacent complementary cones containing it. (It is clearly *in* the intersection. The requirement here is that the intersection contain nothing else.) "Proper" in our sense implies "proper" in Saigal's, but not conversely. For instance, if a full cone is adjacent to a degenerate cone the common face would be considered "proper" by Saigal's definition but not by ours. Hence, our definition of $\mathrm{K}(M)$ being regular is strictly stronger. We will use the italic *proper* and *regular* to refer to Saigal's (1972b) definition, and standard lettering for our own definitions. Notice the definitions are equivalent for nondegenerate matrices. As pointed out before, Saigal (1972b) incorrectly "proves" that $M \in \overline{\mathbf{Z}}$ implies $\mathrm{K}(M)$ is *regular*. An example of a $\overline{\mathbf{Z}}$-matrix where $\mathrm{K}(M)$ is neither regular or *regular* was given in the last section. However, the paper also contained the "theorem" that if $\mathrm{K}(M)$ is *regular*, $M \notin \mathbf{P}$, M is nondegenerate and $\mathrm{sol}(q, M)$ contains only nondegenerate solutions, then $|\mathrm{sol}(q, M)| = 2$. This is also incorrect. For example, letting M be the negative of the identity matrix in $\Re^{2\times 2}$ we have M is nondegenerate, $\mathrm{K}(M)$ is regular and hence is *regular*, $M \notin \mathbf{P}$ and yet $q \in \mathrm{int}\,\mathrm{K}(M)$ implies $|\mathrm{sol}(q, M)| = 4$. A possible substitute here could be gotten from Corollary 4.6 which would state that if M is nondegenerate, $M \notin \mathbf{P}$, $\mathrm{K}(M)$ is regular (so $M \in \mathbf{INS}$ by Corollary 3.18), then if $\mathrm{sol}(q, M)$ contains only nondegenerate solutions, then $|\mathrm{sol}(q, M)|$ is even. These two errors in Saigal (1972b) cause some results of Mohan (1978), which depend on them, to be incorrect. These results are Theorems 1.3.8, 1.5.8, 1.5.12, 3.3.3, 3.3.4, and Corollary 3.3.1 of Mohan (1978).

Aside from questions concerning the exact number of solutions, another concept that has been studied is the constant parity property. We say a matrix M has the *constant parity property* if and only if the parity of $|\mathrm{sol}(q, M)|$, i.e., whether it is even or odd, is the same over all q where $\mathrm{sol}(q, M)$ contains no degenerate solutions. (Thus if $M \notin \mathrm{Q}$ and has the constant parity property then the parity is even. Given any $q \notin \mathrm{K}(M)$ we have $\mathrm{sol}(q, M) = \emptyset$ contains no degenerate solutions and has even parity.) The concept of constant parity is a weaker form of the concept of a constant number of solutions. Clearly all INS-matrices have the constant parity property.

The classic theorem on constant parity was shown by Murty (1972). It states that a nondegenerate matrix has the constant parity property. Also in this paper is the theorem that a nonnegative Q-matrix has the constant parity property with the parity being odd.

In Saigal (1970b) we find the following theorem on the constant parity property: If $-M^T \in \mathrm{S}$, then M has the constant parity property with the parity being even. The final word on the subject was in essence given by Saigal (1972a). It states that a matrix, $M \in \Re^{n \times n}$, has the constant parity property if and only if it is true that for any collection $\operatorname{pos} C(\alpha_1), \operatorname{pos} C(\alpha_2), \ldots,$ $\operatorname{pos} C(\alpha_k)$ of strongly degenerate complementary cones, where k is odd and

$$\dim[\operatorname{pos} C(\alpha_1) \cap \cdots \cap \operatorname{pos} C(\alpha_k)] = n - 1,$$

there exists for each q in this intersection another strongly degenerate complementary cone, $\operatorname{pos} C(\alpha_{k+1})$, such that $q \in \operatorname{pos} C(\alpha_{k+1})$. This result expresses the basic geometric structure behind the constant parity property.

Mohan (1978) proves the following related theorem concerning **Z**-matrices: If $M \in \mathbf{Z} \cap \Re^{n \times n}$ and there is a $x \in \Re^n$ such that $M^T x > 0$, then M has the constant parity property and the parity is odd if and only if $M \in \mathbf{K}$.

The last area of complementarity theory we will bring up is Lemke's algorithm. An algorithm for solving the LCP was suggested by Lemke and Howson (1964), and Lemke (1965). It has since become a major tool in the field, inspiring much research into other algorithms based on the same principles and causing many studies to determine conditions for which the algorithm is guaranteed to "process" a given LCP. For a detailed description of Lemke's algorithm see the two papers mentioned or see Eaves (1971) or Cottle (1983). The essential concept is as follows. Given the LCP (q, M), we take some vector $0 < d \in \Re^n$ and consider the family of LCPs $(q + \theta d, M)$, where the parameter θ is taken as a nonnegative number. (In the canonical statement of the algorithm, d is taken to be $(1, 1, \ldots, 1)^T$.) For all θ large enough we will have $q + \theta d \geq 0$ and hence $(q + \theta d, 0) \in \mathrm{sol}(q + \theta d, M)$. In other words, the "tail" of the ray

$$\{ q + \theta d \mid \theta \geq 0 \} \tag{5.11}$$

is contained in the positive quadrant. We then move back along the ray (5.11) attempting to get to q. When we reach the face of a complementary cone we continue in the adjacent cone. Thus a proper face allows us to travel in the same direction along the ray (5.11) as we had been traveling, while a reflecting face causes us to change direction. The problems associated with reaching a degenerate face, or with reaching a nondegenerate face on its boundary, can be taken care of by lexicographical methods. Again, see Eaves (1971). The actual algorithm is carried out by a pivoting scheme which gives us a solution to the LCP $(q + \theta d, M)$ when we are at the point $q + \theta d$ of the ray (5.11).

(This solution is associated with the complementary cone through which we are currently traveling.) The hope is we eventually reach the end-point of the ray (5.11) and thus find a solution to the original **LCP** (q, M). The other two possiblities are that we go off on the infinite end of the ray (5.11) never to return, or we reach a degenerate face with no other full cone to travel through than the one by which we arrived. It is now clear for $M \in$ INS where $\mathrm{K}(M)$ is star-shaped at $d > 0$ that we will find:

1) Lemke's method using d will process (q, M) for all $q \in \Re^n$.

2) If a solution is found, then θ will have been monotonically decreasing. That is, after the first pivot to initialize the algorithm, each pivot will cause θ to be strictly smaller. (Actually, to prevent degeneracy, we use lexicographical techniques. In this case the vector used in place of θ is lexicographically decreasing.)

3) If when running the algorithm we find that θ increases, or that we reach a degenerate face, we may conclude (q, M) has no solution.

While it is necessary $\mathrm{K}(M)$ be star-shaped at $d > 0$ for these conditions to hold, it is not necessary that M belong to INS. For the matrix (5.10), $\mathrm{K}(M)$ is star-shaped at $d = (1,1)^T$, and the above three conditions hold. However, (5.10) is not in INS.

These observations, stated with a different emphasis, are basically seen in Theorem 4.1 of Saigal (1972b). This theorem states that if $\mathrm{K}(M)$ is *regular* and contains no strongly degenerate cones, then a necessary and sufficient condition for Lemke's algorithm to solve (q, M) using $d > 0$ for all $q \in \mathrm{K}(M)$ is that $\mathrm{K}(M)$ be star-shaped at d. In addition, the theorem states that θ will be monotonically nonincreasing. As pointed out above, we may replace "*regular*" in this theorem by "regular." In this case, the

condition that $\mathrm{K}(M)$ contain no strongly degenerate cones can be dropped. It is interesting to note that the theorem is false in one direction. While the star-shapedness is certainly necessary, it is not sufficient. Let

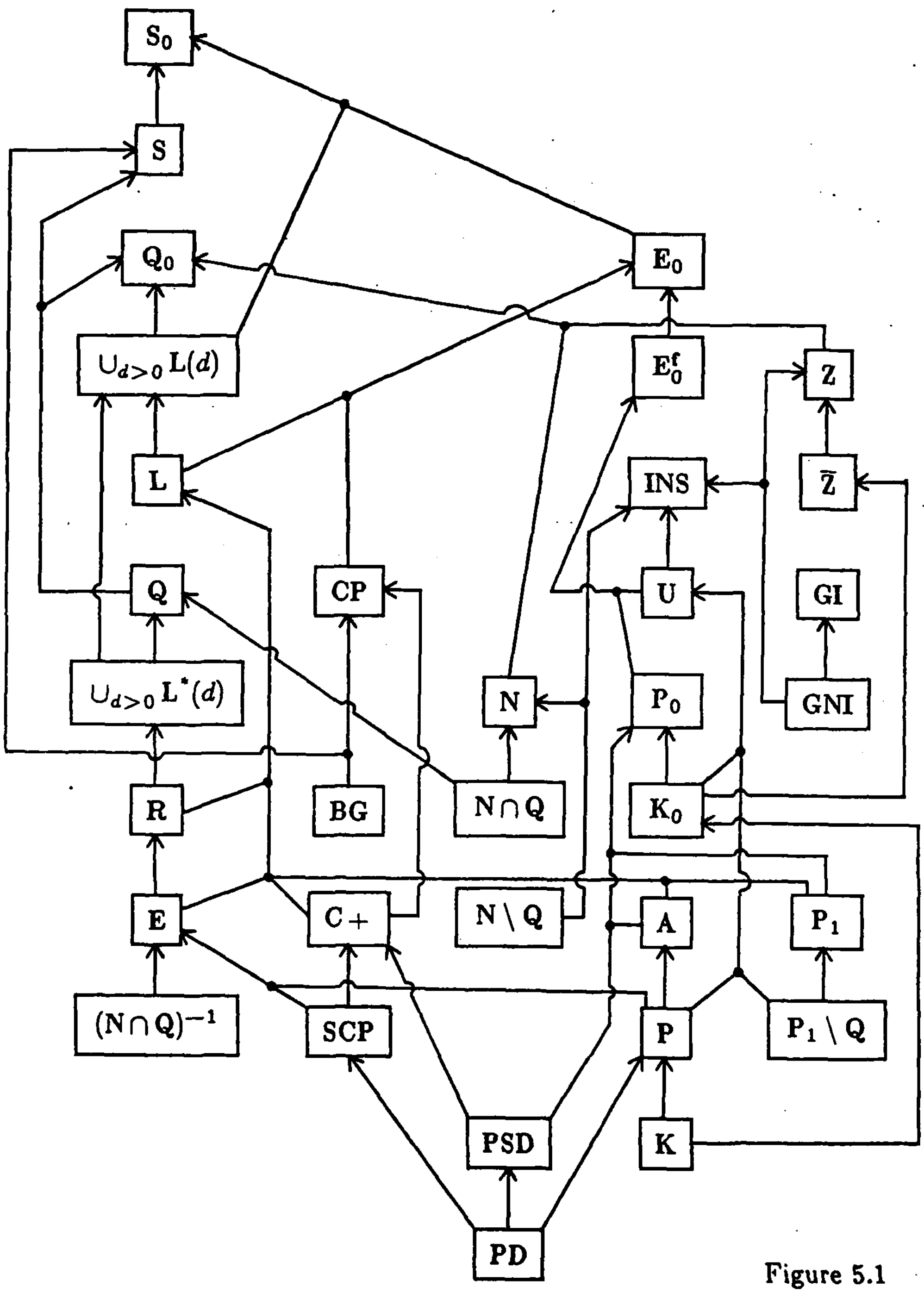
$$M = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

We find $\mathrm{K}(M) = \operatorname{pos} -M$. Also, $\mathrm{K}(M)$ contains no strongly degenerate cones. Notice that all faces of all complementary cones are contained in $\partial\,\mathrm{K}(M)$, except for $\operatorname{pos} C(\emptyset)_{.\hat{2}}$ and $\operatorname{pos} C(\{2\})_{.\hat{1}}$. However, the complementary cones adjacent on $\operatorname{pos} C(\emptyset)_{.\hat{2}}$ are the full cone $\operatorname{pos} C(\emptyset)$ and the degenerate cone $\operatorname{pos} C(\{2\})$. Thus $\operatorname{pos} C(\emptyset)_{.\hat{2}}$ is *proper*, but not proper. Similarly, the cones adjacent on $\operatorname{pos} C(\{2\})_{.\hat{1}}$ are the degenerate cone $\operatorname{pos} C(\{2\})$ and the full cone $\operatorname{pos} C(\{1,2\})$. Thus $\mathrm{K}(M)$ is *regular*, but not regular. It is certainly star-shaped at $d = (1,1,1)^T$. Yet, while $q = (1,-1,2)$ is contained in $\mathrm{K}(M)$, in fact we have $(0,0,0,1,1,2) \in \mathrm{sol}(q,M)$, Lemke's algorithm finds no solution to (q,M) using any $d > 0$. Thus the sufficiency part of Theorem 4.1 in Saigal (1972b) is in error.

One more point before finishing this chapter is that Saigal (1972b) defines $\mathrm{K}(M)$ to be the union of all complementary cones of dimension $n-1$ or greater, where $M \in \Re^{n\times n}$. While this is often the case, it is not always true. For example, let

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now $q = (-1, -1, 0, 0)^T \in \mathrm{K}(M)$. Yet, if $q \in \operatorname{pos} C(\alpha)$ then $\{3, 4\} \subseteq \alpha$ and hence $\dim[\operatorname{pos} C(\alpha)] < 3$. The inclusion relationships of the matrix classes discussed in this chapter are diagrammed in the following figure.



S_0
S
Q_0
E_0
$\cup_{d>0} L(d)$
E_0^f
Z
L
INS
$\bar{Z}$
Q
CP
U
GI
$\cup_{d>0} L^*(d)$
N
P_0
GNI
R
BG
$N \cap Q$
K_0
E
C+
$N \setminus Q$
A
P_1
$(N \cap Q)^{-1}$
SCP
P
$P_1 \setminus Q$
PSD
K
PD

Figure 5.1

CHAPTER 6.
CONCLUSION

The central emphasis of this work has been on the underlying geometric stucture of **LCP**'s with the global property of an invariant number of solutions. There are other interesting open questions related to this, and to **LCP** theory in general. It seems appropriate to mention some of these questions as a conclusion to this study.

Theorem 2.22 shows that $\mathbf{Q}_0 \cap \mathbf{U} \subseteq \mathbf{P}_0$. In essence, if we think of starting in the positive orthant, which is a positive complementary cone, and "moving" in $\mathrm{K}(M)$ through a sequence of adjacent complementary cones then, if $M \in \mathbf{Q}_0 \cap \mathbf{U}$, every common face we encounter between two complementary cones is proper, until we reach a degenerate face which must be on the boundary. Since "reflecting" isn't allowed, as those type of faces are forbidden by the fact that $M \in \mathbf{U}$, and since int $\mathrm{K}(M)$ is path connected so we can reach all the complementary cones, then we can never reach a negative cone. (There isn't enough "room," and there are too many restrictions, to allow us to "turn around.") It seems that there isn't enough "room" even allowing degenerate faces within int $\mathrm{K}(M)$. Thus a problem left open by this study is to determine whether or not $\mathbf{Q}_0 \cap \mathbf{E}_0^{\mathbf{f}} \subseteq \mathbf{P}_0$.

Look once again at the map, F, used in the proof of Theorem 3.15.

If, as before, we assume that no complementary cone of $\mathrm{K}(M)$ is strongly degenerate, then we can associate with F, and hence with M, a special integer referred to as the *degree* of F (of M). Let $q \in \Re^n$ be any vector that is nondegenerate with respect to M. Then the degree of F (of M) is the number of positive complementary cones containing q minus the number of negative complementary cones containing q. (It can be shown that this number will be invariant over all q nondegenerate with respect to M. For more on the concept of *degree* see Ortega and Rheinboldt (1970).) The degree of a map is a measure of the number of points in the domain which are mapped to each point in the range. For a general map of degree k, however, it is *not* necessary that *any* point in the range have exactly $|k|$ points mapping into it from the domain. However, the map F associated with an **LCP** is not a *general* map. Perhaps it is the case for these special maps, that when the degree of F is k, one can always find a point in the range which has exactly $|k|$ points of the domain mapping into it. In the case $k = 0$, this would mean that every matrix M with zero degree is not in **Q**, i.e., some point, q, in the range, $\Re^n$, of F has no point in the domain mapping into it. (Note that q would then trivially be nondegenerate with respect to M.) This is not the case. Kelly and Watson (1979) show that the nondegenerate matrix

$$M = \begin{bmatrix} 21 & 25 & -27 & -36 \\ 7 & 3 & -9 & 36 \\ 12 & 12 & -20 & 0 \\ 4 & 4 & -4 & -8 \end{bmatrix}$$

is in **Q**, yet it is a straightforward calculation to verify that the degree of M is zero. For the case $k \neq 0$ the question is still open, and there are reasons to believe that the geometric structure of non-zero degree matrices is significantly different from the geometric structure of zero degree matrices.

Hence we have the deep question in **LCP** theory of determining whether or not there exists a matrix M with no strongly degenerate complementary cones, and with degree $k \neq 0$, such that $|\mathrm{sol}(q, M)| > |k|$ whenever q is nondegenerate with respect to M. Another way of phrasing this is to ask if, for matrices M with no strongly degenerate complementary cones, it is true that when the union of the positive complementary cones is $\Re^n$ *and* the union of the negative complementary cones is $\Re^n$, then every vector q, nondegenerate with respect to M, is contained in the same number of positive complementary cones as negative complementary cones. Indeed, this is conjectured to be true in Garcia and Gould (1980). See also Howe (1980). (It should be pointed out that the class "$\mathbf{Q_0}$" in Garcia and Gould (1980) is not the same as the one discussed in the present work.)

At the end of Chapter 3 we showed that, for nondegenerate matrices M, $\mathrm{K}(M)$ is regular if and only if $M \in \mathbf{INS}$. It seems that the nondegeneracy assumption should be unnecessary; this raises the question of whether it is in general true that $\mathrm{K}(M)$ is regular if and only if $M \in \mathbf{INS}$.

In Chapters 3 and 4 we developed the idea of the partition Σ of $\Re^n \setminus K(M)$. We noted that the elements of Σ are *not* in general convex, not even when considering only those elements contained in $\mathrm{K}(M)$. The question then arises as to whether the elements of Σ are, in general, star-shaped. This question is open, as is the related question of whether there will always be an element of Σ which *is* convex. (Is Theorem 4.4 valid for degenerate matrices as well as nondegenerate matrices?)

In Chapter 4 we already have discussed Conjecture 4.8, but have not spoken about Assumption 4.17. This is a technical assumption that has been used in another form by other authors. The last open question we'll

consider is the one surrounding this assumption on the geometry of **LCP**'s. It can be stated as follows. Given an **LCP** consider the related map F as defined in the proof of Theorem 3.15. Suppose we let D be the union of some collection of orthants in $\Re^n$ such that D forms a pseudomanifold, i.e., between any two orthants in D there is a path, in D, of "neighboring" orthants. The image under F of each orthant is a complementary cone. If the complementary cones which are the images of the orthants in D are all positive complementary cones is it then the case that the restricted map $F : D \rightarrow \Re^n$ is injective? If $D = \Re^n$ the answer is "yes" as shown in Murty (1972). If D is convex we can reduce the problem to the case where D is $\Re^m$ for some $m \leq n$ and the answer is again "yes" by the result in Murty (1972). In general the question is open. It should be noted that the **LCP** structure is important. If we were to require the function F on D to just be piecewise-linear, with the pieces of linearity being the orthants, and the determinants of the matrices defining the affine functions on *adjacent* orthants to be of opposite signs, then $F : D \rightarrow \Re^n$ would not necessarily be injective. As an example, consider $D = \Re^3$, and F defined as follows on the different orthants

$$F(x_1, x_2, x_3) = \begin{cases} (x_1, x_2, x_3) & \text{if } x_1, x_2, x_3 \geq 0 \\ (x_1, x_1 + x_2, x_1 + x_3) & \text{if } x_1 \leq 0, x_2 \geq 0, x_3 \geq 0 \\ (x_1 + x_2, x_2, x_2 + x_3) & \text{if } x_1 \geq 0, x_2 \leq 0, x_3 \geq 0 \\ (x_1 + x_3, x_2 + x_3, x_3) & \text{if } x_1 \geq 0, x_2 \geq 0, x_3 \leq 0 \\ F(-x) & \text{otherwise} \end{cases}$$

Then F is not injective even though it satisfies all the other restrictions mentioned.

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